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First-order Ordinary Differential Equations, Symmetries and Linear Transformations

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We present an algorithm for solving first-order ordinary differential equations, by systematically determining symmetries of the form $[\xi = F(x),\, \eta = P(x)\, y + Q(x)]$, where $\xi\, \partial/\partial x + \eta\, \partial/\partial y$ is the symmetry generator. To these *linear* symmetries one can associate an ordinary differential equation class which embraces all first-order equations mappable into separable ones through linear transformations $\{t = f(x),\, u = p(x)\, y + q(x)\}$. This single class includes as members, for instance, 429 of the 552 solvable first-order examples of Kamke's book. Concerning the solution of this class, a restriction on the algorithm being presented exists, only in the case of Riccati equations, for which linear symmetries *always* exist, but the algorithm will only partially succeed in finding them.

1 Introduction

One of the most attractive aspects of Lie's method of symmetries is its generality: roughly speaking, all solution methods for differential equations can be correlated to particular forms of the symmetry generators [2, 16]. However, for first-order ordinary differential equations, Lie's method seems to be, in principle, not as useful as in the higher order case. The problem is that the determining partial differential equation — whose solution gives the infinitesimals of the symmetry group — has the original first-order equation in its characteristic strip. Hence, finding these infinitesimals requires solving the original equation, which in turn is what we want to solve using these infinitesimals, thus invalidating the approach.

For higher order ordinary differential equations, the strategy consists of restricting the cases handled to the universe of equations having *point* symmetries, so that the infinitesimals depend on just two variables, and then the determining partial differential equation is overdetermined. Although few second or higher order equations have point symmetries, and the solution of the corresponding partial differential equation system for the infinitesimals may be a major problem in itself [9], the hope is that one will be able to solve the system by taking advantage of the fact that it is overdetermined.

One basic motivation in this approach is also that the finite transformations associated with point symmetries are pointlike and these transformations form a group *by themselves* — not just with respect to the Lie group parameter. Indeed, the composition of *any* two point transformations is also a point transformation. Consequently any two point symmetries can be obtained from each other through a point transformation. Lie point symmetries can then be used to tackle the *equation class*, all of whose members can be obtained from (are equivalent to) each other through point transformations, and which includes a member we know how to solve (missing the dependent variable).

However, such a powerful approach is not useful in the case of first-order equations — the subject of this paper — for which "point symmetries" are already the most general ones. The alternatives left then, roughly speaking, consist of: looking for particular solutions to the determining partial differential equation [4], or restricting the form of the infinitesimals trying to emulate what is done in the higher-order case, so that the problem can be formulated in terms of an *overdetermined* linear partial differential equation system [5, 10]. The question in this latter approach, however, is what would be an "appropriate restriction" on the symmetries such that:

- the related invariant equation family includes a reasonable variety of non-trivial cases typically arising in mathematical physics;
- the determination of these symmetries, when they exist, can be performed systematically, preferably without solving any differential equations;
- the related finite transformations form a group by themselves — not just with respect to the Lie group parameter — so that the method applies to a whole equation class.

Bearing this in mind, this paper is concerned with first-order ordinary differential equations and *linear symmetries* of the form

$$\xi = F(x), \quad \eta = P(x)\, y + Q(x) \tag{1.1}$$

where $\{\xi,\ \eta\}$ are the infinitesimals, the symmetry generator is $\xi\,\partial/\partial x + \eta\,\partial/\partial y$ and x and $y \equiv y(x)$ are respectively the independent and dependent variables. Concerning the arbitrary functions $\{F,\ P,\ Q\}$, the requirements are those implied by the fact that (1.1) generates a Lie group of transformations. The linear symmetries (1.1) have interesting features; for instance, the related finite transformations are also linear, of the form

$$t = f(x), \quad u = p(x)\, y + q(x), \tag{1.2}$$

where t and $u \equiv u(t)$ are respectively the new independent and dependent variables, and $\{f,\ p,\ q\}$ are arbitrary functions of x. Linear transformations (1.2) form a group by themselves too, not just with respect to the Lie parameter. So, as it happens with point symmetries in the higher order case, *any* two linear symmetries (1.1) can be transformed into each other by means of a linear transformation (1.2), and hence we can associate an equation class with the symmetries (1.1). Since separable equations have symmetries of this form, the class of equations admitting linear symmetries (1.1) actually includes all first-order equations which can be mapped into separable ones by means of linear transformations (1.2).

We note that in the particular case of polynomial equations, e.g. of Abel type[1],

$$y' = f_3(x)\,y^3 + f_2(x)\,y^2 + f_1(x)\,y + f_0(x), \tag{1.3}$$

where f_3, f_2, f_1 and f_0 are arbitrary functions of x, (1.2) actually defines their respective classes[2]. Since a separable Abel equation can be obtained by just taking the coefficients f_i all equal, this means that there are complete Abel classes all of whose members can be transformed into separable ones by means of (1.2). Such a case was discussed and solved at the end of the nineteenth century by Liouville, Appell and others and is presented in textbooks such as [11, 13].

More generally, for polynomial equations of the form

$$y' = f_n(x)\,y^n + f_1(x)\,y + f_0(x), \tag{1.4}$$

where f_n, f_1 and f_0 are arbitrary functions x, Chini [7, 11] presented a method similar to this mapping into separable equations, but through transformations (1.2) with $q = 0$. Chini's method is equivalent to solving (1.4) by determining, when they exist, symmetries (1.1) with $Q = 0$.

In connection with the above, this work presents a generalization of these methods as an algorithm for determining whether or not a first order ordinary differential equation of arbitrary form[3]

$$y' = \Phi(x, y), \tag{1.5}$$

belongs to this class admitting linear symmetries (1.1), and, if so, for explicitly finding the symmetry, without restrictions to the form of $\{F,\ P,\ Q\}$. Both the determination of the existence of a solution as well as of the symmetry itself are performed without solving any auxiliary differential equations.

The exposition is organized as follows. In sec. 2, the connection between the symmetries of the form (1.1) and linear transformations of the form (1.2) is analyzed and a solution algorithm for the related class is presented. Some examples illustrating the type of problem which can be solved using this method are shown in sec. 2.2. In sec. 3, a discussion of Riccati equations in terms of their symmetries and of a variant of the method of sec. 2, to solve a subset of the Riccati problem, is given. In sec. 4, a discussion and some statistics are presented concerning the classification of Kamke's first-order ordinary differential equation examples. Finally, the conclusions contain general remarks about this work.

2 Linear transformations and symmetries

To determine whether or not a given first-order equation has symmetries of the form (1.1), following [5], we take advantage of the fact that, for such a symmetry, the related invariant equation family can be computed in closed form. With the invariant family in

[1] In what follows we use $y' = \frac{dy}{dx}$.

[2] The Abel equations members of a given class can be mapped between themselves through (1.2). There are infinitely many non-intersecting such Abel classes but only some of them (still infinitely many) admit symmetries of the form (1.1).

[3] Riccati equations are partially excluded from the discussion; see sec. 3.

hand, we then show how one can algorithmically compute the infinitesimals (1.1), when they exist, by quadratures. For that purpose, with no loss of generality, we first rewrite (1.1) in terms of $\{f(x), p(x), q(x)\}$ (see (1.2)) as[4]

$$\xi = \frac{1}{f'}, \quad \eta = -\frac{p'\,y + q'}{f'\,p}. \tag{2.1}$$

A direct computation of the finite transformations generated by (2.1) shows that they are linear, of the form

$$t = z(x), \quad u = \frac{p(x)\,y + q(x) - q(z(x))}{p(z(x))} \tag{2.2}$$

where $z(x)$ is a solution of $f(z) - f(x) - \alpha = 0$ and α is the (Lie) group parameter.

The invariant equation family related to (2.1) can be obtained, for instance, by computing the differential invariants $\{I_0,\, I_1\}$ of order 0 and 1 related to (2.1):

$$I_0 = py + q, \quad I_1 = \frac{f'}{p'\,y + py' + q'}, \tag{2.3}$$

and hence the invariant equation, given by $\Lambda(I_0, I_1) = 0$, where Λ is arbitrary, can be conveniently written as $I_1 G(I_0) = 1$, with arbitrary G, resulting in

$$y' = \frac{f'}{p} G(py + q) - \frac{q'}{p} - \frac{p'}{p} y. \tag{2.4}$$

Due to this connection between linear transformations and symmetries (1.1), the same invariant family (2.4) can be obtained directly from an autonomous equation,

$$u' = G(u) \tag{2.5}$$

by just changing variables in it using (1.2). The solution to (2.4) can be obtained in the same way, by changing variables in the solution to (2.5):

$$f - \int^{py+q} \frac{1}{G(z)} dz + C_1 = 0 \tag{2.6}$$

In fact (2.1) can also be obtained from the symmetry $[\xi = 1,\, \eta = 0]$ of (2.5) by changing variables using (1.2). We recall that the knowledge of $\{\xi,\, \eta\}$ in (2.1) suffices to express the solution of any member of the class (2.4) by quadratures and so it is equivalent to the determination of $\{f,\, p,\, q\}$ in (2.6).

Theorem 1 *Consider $y' = \Psi(x, y)$ with $\Psi_{yyy} \neq 0$. The determination of whether or not this equation belongs to the class (2.4), and, when it does, the computation of the infinitesimals (2.1) themselves, can be performed algorithmically from Ψ by quadratures.*

Proof. We start by noting an intrinsic feature of equations that are members of the

[4] In this section we assume $f' \neq 0$, since, otherwise, (2.4) would be a first-order linear equation. The limiting case where $f' \to \infty$, $\xi = 0$, however, is not excluded.

class (2.4): the determination of $p(x)$ up to a constant factor, say κ, suffices to map any member of the class into another one having a symmetry of the form

$$\xi = F(x), \quad \eta = Q(x). \tag{2.7}$$

This is easily verified by changing variables

$$y = \frac{u}{\kappa\, p} \tag{2.8}$$

in (2.1), arriving at $[\xi = 1/f',\ \eta = -q'\kappa/f']$, which is of the form (2.7). In turn, symmetries of the form (2.7), when they exist, can be systematically determined as shown in [5]. In what follows, we develop the proof by first showing how to map any equation member of (2.4) into one having a symmetry of the form (2.7), and then, for completeness, briefly reviewing how such a symmetry is determined when it exists.

For all equations $y' = \Psi$ of the class (2.4), we have that

$$\begin{array}{lcl} \Psi_y & = & f'G'(py+q) - \frac{p'}{p}, \\ \Psi_{yy} & = & f'pG''(py+q), \\ \Psi_{yyy} & = & f'p^2G'''(py+q). \end{array} \tag{2.9}$$

Thus we let

$$A \equiv \frac{\Psi_{yy}}{\Psi_{yyy}} = \frac{1}{p} K(py+q) \quad \text{where} \quad K = \frac{G''}{G'''}. \tag{2.10}$$

Three cases now arise, related respectively to whether $A_y = 0$, $A_{yy} = 0$ or $A_{yy} \neq 0$.

Case 2.1: $A_y = 0$

In this case, $K' = 0$, so that $K = \kappa$ for some non-zero κ, and hence $A = \kappa/p$. So, from (2.8), when $y' = \Psi$ is indeed a member of the class (2.4), by changing variables using

$$y = A\, u, \tag{2.11}$$

the resulting equation family will have a symmetry of the form (2.7).

Case 2.2: $A_{yy} = 0,\ A_y \neq 0$

In this case, $K'' = 0$, so that from (2.10)

$$A = \frac{\kappa_1\,(p\,y+q) + \kappa_0}{p} \tag{2.12}$$

for some constant κ_0 and some non-zero constant κ_1. Here the necessary condition for $y' = \Psi$ to be a member of (2.4) is that the ratio above be linear in y. In such a case, when the equation is indeed a member of this class, by introducing

$$u = \ln(A), \tag{2.13}$$

the resulting equation in u will have a symmetry of the form (2.7); this can be verified straightforwardly by performing the change of variables directly in (2.1).

Case 2.3: $A_{yy} \neq 0$

In this case let

$$I \equiv \frac{A_{yx}}{A_{yy}} = \frac{p'y + q'}{p}. \tag{2.14}$$

The necessary condition for $y' = \Psi$ to be a member of (2.4) is then that I be linear in y, whence

$$p(x) = \exp\left(\int I_y dx\right). \tag{2.15}$$

So, when the equation belongs to this class, from (2.8), changing variables $y = u/p$ will lead to an equation having a symmetry of the form (2.7).

Once we have shown how a member of (2.4) can be mapped into another one having a symmetry of the form (2.7), what remains to be done in the proof of Theorem 1 is to review how that symmetry can be obtained by quadratures.

2.1 Symmetries of the form $[\xi = F(x),\ \eta = Q(x)]$

By computing differential invariants, as we did to arrive at (2.4), the invariant equation family associated to $[\xi = F(x),\ \eta = Q(x)]$ can be written as[5]

$$y' = \Phi(x, y) \equiv \frac{1}{F(x)}\left(Q(x) + G\left(y - \int \frac{Q(x)}{F(x)} dx\right)\right), \tag{2.16}$$

where F, Q and G are arbitrary functions of their arguments. So far we have shown that if an equation belongs to (2.4), then after changing variables as shown for the cases 2.1-3, the resulting equation will be a member of this family (2.16).

Now, to determine F and Q, following [5], we first construct an expression depending on x and y only through G,

$$\mathcal{K} \equiv \frac{\Phi_y}{\Phi_{yy}} = \frac{G_y}{G_{yy}}, \tag{2.17}$$

where we assume[6] $\Phi_{yy} \neq 0$. As explained in [5], the problem then splits into two cases.

Case 2.4: $\mathcal{K}_y \neq 0$

In this case, we can obtain the ratio $Q(x)/F(x)$, only depending on x, by taking

$$\Upsilon \equiv \frac{\mathcal{K}_x}{\mathcal{K}_y} = -\frac{Q(x)}{F(x)}. \tag{2.18}$$

The knowledge of this ratio in turn permits the elimination of Q from the determining partial differential equation for the infinitesimals, leading to[7]

[5] If $Q = 0$ or $F = 0$, then the invariant equation is separable or linear, so in (2.16) and henceforth we assume $F \neq 0$, $Q \neq 0$.

[6] If $\Phi_{yy} = 0$, then (2.16) is already a first-order linear equation solvable in terms of quadratures.

[7] For more details see [5].

$$F(x) = C_1 \exp\left(\int \left(\frac{\Upsilon\,\Phi_y - \Upsilon_x - \Phi_x}{\Phi + \Upsilon}\right) dx\right), \tag{2.19}$$

which, together with (2.18), gives the solution we are looking for. The necessary and sufficient conditions for the existence of such a symmetry are

$$\frac{\partial}{\partial y}\left(\frac{\mathcal{K}_x}{\mathcal{K}_y}\right) = 0, \qquad \frac{\partial}{\partial y}\left(\frac{\Upsilon\,\Phi_y - \Upsilon_x - \Phi_x}{\Phi + \Upsilon}\right) = 0. \tag{2.20}$$

Case 2.5: $\mathcal{K}_y = 0$

Since $\mathcal{K}_y = -\mathcal{K}_x\,F/Q$, then, when $\mathcal{K}_y$ is zero, $\mathcal{K}_x$ also vanishes, so $\mathcal{K} = \kappa$ for some non-zero constant κ. Hence, the right-hand side of (2.16) satisfies

$$\frac{\Phi_y}{\Phi_{yy}} = \kappa \tag{2.21}$$

and so (2.16), the invariant equation family, is of the form

$$y' = \Phi \equiv A(x) + B(x)\,\mathrm{e}^{y/\kappa} \tag{2.22}$$

where A and B are arbitrary functions. For a given equation of this type, A and B can be determined by inspection, and the determining partial differential equation for the infinitesimals can be solved directly in terms of A and B as

$$F(x) = \frac{\exp\left(-\int \frac{A}{\kappa} dx\right)}{B}, \qquad Q(x) = A\,F(x). \tag{2.23}$$

2.2 Examples

Example 2.1. Consider the first-order equation, example 128 from Kamke's book,

$$xy' + ay - f(x)g(x^a y) = 0, \tag{2.24}$$

where a is an arbitrary constant and f and g are arbitrary functions of their arguments. For this equation, Kamke shows a change of variables mapping the equation into a separable one, derived for this particular equation family in [8]. Using the algorithm presented in this paper, we tackle this equation by computing A in (2.10):

$$A = \frac{g''}{x^a\,g'''}, \tag{2.25}$$

so we are in Case 2.4. We then proceed by computing I (see (2.14)) arriving at

$$I = \frac{a\,y}{x}. \tag{2.26}$$

The existence condition that I be linear in y is satisfied; hence, according to (2.15),

$$p(x) = x^a. \tag{2.27}$$

Now, changing variables $y = \dfrac{u}{p}$ as indicated in (2.8), (2.24) becomes

$$u' = g(u)\,\frac{f(x)x^a}{x}, \tag{2.28}$$

which is already separable (and thus naturally has a symmetry of the form (2.7)). The solution to (2.24) is then obtained by changing variables back in the solution to (2.28), leading to

$$\int x^{a-1}\, f(x)\, dx - \int^{x^a y} \frac{1}{g(z)} dz - C_1 = 0. \tag{2.29}$$

Example 2.2. We now discuss an example for which $A_{yy} = 0$,

$$y' = \left(x^3y^4 + 4\,x^4y^3 + 6\,x^5y^2 + 4\,x^6y + x^7\right)(x^a + 1) - \frac{y}{x} - 2, \tag{2.30}$$

where a is an arbitrary constant. For this equation, from (2.10),

$$A = \frac{y+x}{2} \tag{2.31}$$

so the change of variables here is $u = \ln(A)$ (see (2.13)), mapping (2.30) into

$$u' = t^{7+a}\,(t^a + 1)\,e^{6\,u} + \frac{7}{t}. \tag{2.32}$$

For this equation, a symmetry of the form (2.7) is computed algorithmically (see sec. 2.1):

$$\xi = \frac{1}{8\,(t^a+1)}, \quad \eta = -\frac{1}{8\,t\,(t^a+1)}. \tag{2.33}$$

Changing variables back directly in the above we arrive at a symmetry for (2.30),

$$\xi = \frac{1}{8\,(x^a+1)}, \quad \eta = -\frac{y+2\,x}{8\,x\,(x^a+1)}, \tag{2.34}$$

whence an implicit solution to (2.30) follows as

$$x + \frac{1}{3\,x^3\,(y+x)^3} + \frac{x^{(1+a)}}{1+a} = C_1. \tag{2.35}$$

Example 2.3. As an example of the case in which $A_y = 0$, consider

$$y' = b\,e^{a\,x\,y}x^a + \frac{\left(x^2-1\right)y}{x} - \frac{1}{x^2} + \ln(x) + c, \tag{2.36}$$

where a, b and c are arbitrary constants. From (2.10), $A = 1/(a\,x)$, so that by changing variables as indicated in (2.11), (2.29) becomes

$$u' = ut + a\,b\,t^{(a+1)}e^u - \frac{a}{t} + a\,t\,(\ln(t) + c)\,. \tag{2.37}$$

This equation has a symmetry of the form (2.7),

$$\xi = \frac{1}{t}, \;\; \eta = -\frac{a}{t^2}, \tag{2.38}$$

whence (2.36) admits the symmetry

$$\xi = \frac{1}{x}, \quad \eta = -\frac{x\,y+1}{x^3}, \tag{2.39}$$

which suffices to integrate (2.36) by either using canonical coordinates or computing an integrating factor.

Example 2.4. The algorithm presented is applicable to higher degree equations too (see Table 1. in sec. 4), provided that it is possible to solve the given equation for y'. Consider, for instance, Kamke's example 394,

$$(y')^2 + 2\,f\,y\,y' + g\,y^2 + \left(f^2 - g\right)\exp\left(-2\int_a^x f(z)dz\right) = 0, \tag{2.40}$$

where a is an arbitrary constant and $f \equiv f(x)$ and $g \equiv g(x)$ are arbitrary functions. For this problem Kamke presents a particular change of variables derived in [14]. To tackle this example using our algorithm we first solve the equation for y':

$$y' = -f\,y \pm \sqrt{\left(y^2 - \exp\left(-2\int_a^x f(z)dz\right)\right)(f^2 - g)}. \tag{2.41}$$

Now, by taking either branch of (2.41), and computing the second derivative of (2.10), we find

$$A_{yy} = \frac{2\exp\left(-2\int_a^x f(z)dz\right)}{3\,y^3}, \tag{2.42}$$

so that I in (2.14) is given by

$$I = f\,y, \tag{2.43}$$

whence we compute p using (2.15). We finally arrive at the symmetry

$$\xi = \frac{1}{\sqrt{f^2 - g}}, \quad \eta = -\frac{f\,y}{\sqrt{f^2 - g}}, \tag{2.44}$$

actually admitted by both branches of (2.41).

These four examples are straightforward problems for the single solving algorithm presented, but we are not aware of any other algorithm for tackling examples like 2 or 3; also, for examples 1 and 4, the changes of variables presented in Kamke are non-obvious and presented in connection with different problems [8, 14]. Despite the presence of arbitrary functions and parameters, both Kamke's examples 128 and 394 are actually particular cases of the class represented by (2.4).

3 Riccati equations

The case of Riccati-type equations

$$y' = f_2\,y^2 + f_1\,y + f_0 \tag{3.1}$$

where $f_i \equiv f_i(x)$, $f_2 \neq 0$ and $f_0 \neq 0$, deserves a separate discussion. All Riccati equations admit symmetries of the form (1.1) and so all of them can be mapped into separable ones using transformations of the form (1.2). However, it is easy to verify that to find such a transformation requires solving the Riccati equation itself. The algorithm of the previous section, which does not rely on solving auxiliary differential equations, only works when $\Psi_{yyy} \neq 0$ (see Theorem 1). The usual approach for solving (3.1) then consists of converting it to a linear second-order ordinary differential equation and using the various methods available. Nonetheless, there are entire subfamilies of (3.1) for which symmetries of the form (1.1) can be found following an approach such as the one presented in the previous section, without using techniques for linear second-order equations. Such an approach is interesting since it enriches the algorithms available for tackling the problem and could be of use for solving some linear equations by mapping them into Riccati ones. For the purpose of discussing these cases, and without loss of generality, we rewrite (2.1) by redefining $f' \rightarrow f/p$:

$$\xi = \frac{p}{f}, \;\; \eta = -\frac{p'\,y + q'}{f}. \tag{3.2}$$

If now, in (2.4), we redefine f' in the same way and take G as a quadratic mapping depending on two arbitrary constants a and b,

$$G = u \mapsto u^2 + a\,u + b \tag{3.3}$$

we arrive at the form of an arbitrary Riccati equation, as general as (3.1), but expressed in terms of these two constants a and b and the functions $\{f,\,p,\,q\}$ appearing in its symmetry generator (3.2):

$$y' = f\,y^2 + \frac{(a+2\,q)\,f - p'}{p}\,y + \frac{((a+q)\,q+b)\,f - q'\,p}{p^2}. \tag{3.4}$$

<u>Case 3.1: $p' = 0$</u>

A first solvable case happens when $p' = 0$, so that in (3.2) both infinitesimals depend only on x and hence the symmetry can be systematically determined as explained in sec. 2.1.

<u>Case 3.2: $q' = 0$</u>

A second solvable case happens when, in (3.4), $q' = 0$, so that the infinitesimals (3.2) are of the form

$$\xi = \mathcal{F}(x), \;\; \eta = \mathcal{P}(x)\,y. \tag{3.5}$$

An algorithm for solving such an equation was presented by Chini [7]. In the case of Riccati equations, Chini's algorithm can be summarized as "to check for the constant character" of the expression[8]

$$\mathcal{I} \equiv \frac{\left({f_0}'f_2 - f_0{f_2}' - 2\,f_0 f_1 f_2\right)^2}{\left(f_0 f_2\right)^3}, \tag{3.6}$$

where f_i are the coefficients of y in (3.1). Whenever $\mathcal{I}$ is constant, the problem is systematically solvable in terms of quadratures (see for instance [11], p.303). Concerning (3.4) when $q' = 0$, a direct computation of $\mathcal{I}$ confirms that in such a case $\mathcal{I}$ is constant. Conversely, another direct computation shows that whenever $\mathcal{I}$ is constant, the equation

[8] This connection between the constant character of an expression like (3.6), constructed with the coefficients of a polynomial equation of the form (1.4), and symmetries of the form (3.5), is valid not only for Riccati equations but for equations of the form (1.4) in general.

will have a symmetry of the form (3.5). To check this, it suffices to solve (3.6) for f_1 and substitute the result into (3.1); the resulting Riccati equation will admit the symmetry

$$\xi = \frac{1}{f_2}\sqrt{\frac{f_2}{f_0}}, \quad \eta = \frac{\left({f_0}' f_2 - f_0 {f_2}'\right)}{2\, {f_0}^2 f_2 \sqrt{\frac{f_2}{f_0}}}\, y, \tag{3.7}$$

which is of the form (3.5).

We can also see the equation class solved by Chini's algorithm, as well as explain the previous results, by noticing that (3.6) is an absolute invariant for (3.1) under transformations of the form

$$t = \tilde{f}(x), \quad u = \tilde{p}(x)\, y \tag{3.8}$$

that is, of the form (1.2) with $q = 0$. In fact, (3.6) can be written as

$$\mathcal{I} = \frac{{s_3}^2}{{s_2}^3} \tag{3.9}$$

where

$$s_2 = f_0 f_2, \qquad s_3 = {f_0}' f_2 - {f_2}' f_0 - 2\, f_0 f_1 f_2 \tag{3.10}$$

are relative invariants of weight 2 and 3 with respect to transformations (3.8) [3].

In summary, Chini's algorithm solves the equation class generated by changing variables (3.8) in the general Riccati equation (3.4) at $q' = 0$, all of whose members have $\mathcal{I} = constant$.

Three additional solvable Riccati families, where the invariant $\mathcal{I}$ is non-constant, are obtained by equating in (3.2) any two of the three arbitrary functions $\{f,\, p,\, q\}$ with $p' \neq 0$ and $q' \neq 0$.

Case 3.3: $f = p$

When a given Riccati equation belongs to this family, then, by changing variables via[9]

$$y = \frac{u}{f} \tag{3.11}$$

in the given equation and in the general form of its symmetry (3.2), we see that the resulting equation in u will admit the symmetry

$$\xi = 1, \;\; \eta = -q', \tag{3.12}$$

which can be determined as explained in sec. 2.1.

Case 3.4: $q = p$

When a given Riccati equation belongs to this family, then, by changing variables via

$$y = u - 1 \tag{3.13}$$

in the given equation and in (3.2), we see that the resulting equation in u will admit a symmetry of the form

[9] f is the coefficient of y^2 in the given equation.

$$\xi = \frac{p}{f}, \;\eta = -\frac{p'}{f}\,u; \tag{3.14}$$

that is, infinitesimals of the form (3.5), and hence the equation will be solvable using Chini's method.

Case 3.5: $f = q$

From (3.4), the Riccati family corresponding to this case is given by

$$y' = f y^2 + \frac{(f\,(a + 2\,f) - p')}{p}\,y + \frac{((a+f)f + b)\,f - f'\,p}{p^2}. \tag{3.15}$$

From (3.2), this equation family admits the symmetry

$$\xi = \frac{p}{f}, \;\eta = -\frac{p'\,y + f'}{f}. \tag{3.16}$$

We have not found an obvious transformation of the form $y = P\,u + Q$ to map this symmetry to one of the forms (2.7) or (3.5). A possible approach would be to directly set up the determining partial differential equation

$$\eta_x + \left(\eta_y - \xi'\right)\left(f_2 y^2 + f_1 y + f_0\right) - \xi\,\left({f_2}' y^2 + {f_1}' y + {f_0}'\right) - \eta\,\left(2\,f_2 y + f_1\right) = 0$$

for the coefficients ξ and η of the infinitesimal generator $\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$ of an arbitrary Riccati equation (3.1). Then take ξ and η as given by (3.16) and run a differential elimination process solving for p. That approach works in principle, but the resulting symbolic expressions are large enough to become untractable even with simple examples.

An alternative approach leading to more tractable expressions is based on using the information we have in (3.15) concerning the existence of two constants a and b. So, by equating the coefficients of (3.15) with those of (3.1), we arrive at the system

$$\begin{gathered} f_2 - f = 0, \\ f_1 - \frac{f_2\,(a + 2\,f_2) - p'}{p} = 0, \quad f_0 - \frac{f_2\left(f_2 a + {f_2}^2 + b\right) - {f_2}'\,p}{p^2} = 0. \end{gathered} \tag{3.17}$$

This system can be solved for a and b, so that, when a Riccati equation belongs to this family, the following two expressions formed from its coefficients f_i will be constants:

$$\begin{aligned} a &= \frac{f_1\,p - 2\,{f_2}^2 + p'}{f_2}, \\ b &= \frac{f_0\,p^2 + f_2\left({f_2}^2 - f_1 p - p'\right) + {f_2}'\,p}{f_2}. \end{aligned} \tag{3.18}$$

At this point, however, we cannot verify the constant character of these expressions because p is still unknown. An expression for p can be obtained by computing the integrability conditions for f_2 and p implied by (3.17):

$$f_2{''} = f_1 f_2{'} - 2\, f_0 p' - p f_0{'} + \frac{\left(f_2{'}\right)^2 + p f_0 f_2{'}}{f_2}, \tag{3.19}$$

$$p'' = 2\, f_2{'} f_2 - p'\, f_1 - p f_1{'} + \frac{f_2{'}\, p' + p f_1 f_2{'}}{f_2}. \tag{3.20}$$

Using (3.19) to eliminate p' from (3.20) we arrive at a solution for p:

$$p = \tag{3.21}$$

$$\frac{f_2 \left(3 s_2\, f_2{''} s_2{'} - 2 {s_2}^2 f_2{'''} + \left(\left(s_2{''} - s_4\right) s_2 - 8 {s_2}^3 - 2\left(s_2{'}\right)^2 + 2 {s_3}^2\right) f_2{'}\right)}{s_2 \left(2\, s_2 s_3{'} - 3\, s_3 s_2{'}\right)},$$

where

$$s_4 = \frac{2\, s_2\, s_3{'} - 3\, s_3\, s_2{'} + 3\, {s_3}^2}{2\, s_2} \tag{3.22}$$

is the relative invariant of weight 4 for Riccati equations with respect to transformations of the form (3.8) [3]. We note that, from (3.9), when the denominator of (3.21) is zero the equation has a *constant* invariant $\mathcal{I}$ (3.6) and so it is already solvable using Chini's algorithm.

In summary, a strategy, not relying on solving second-order linear equations, for finding linear symmetries of the form (1.1) for Riccati equations, consists of

(1) check if the equation has a symmetry of the form $[\xi = \mathcal{F}(x),\ \eta = \mathcal{Q}(x)]$ (algorithm of [5] (see sec. 2.1)); or $[\xi = \mathcal{F}(x),\ \eta = \mathcal{P}(x)\, u]$ (Chini's algorithm);

(2) check if the equation belongs to one of the two families "$f = p$" or "$p = q$" by using the transformations (3.11) and (3.13) and re-entering the previous step;

(3) check if the equation belongs to the family "$f = q$" (3.15); for that purpose:

   (a) compute the invariants (3.10);

   (b) use these invariants to compute p using (3.21);

   (c) plug the resulting p into (3.18) and verify if the two right-hand-sides are constant. If so, the equation admits the symmetry (3.16)

It is our belief that, with the development of computer algebra software and faster computers, these types of algorithms for Riccati subclasses will become more relevant, and they will also provide an alternative for tackling second-order linear equations.

4 Classification of Kamke's examples and discussion

We have prepared a computer algebra prototype of the algorithms presented in sec. 2 and sec. 3 using the Maple system. We have then used this prototype to analyze the set of Kamke's 576 first-order equation examples.

In order to perform the classification, we first excluded from the 576 Kamke examples all those for which a solution is not shown and for which we were not able to determine it by other means[10]. So our testing arena starts with 552 equations.

We know that all equations of type separable, linear, homogeneous, Bernoulli, Riccati and Abel with constant invariant,[11] that is, 372 of Kamke's examples, have symmetries of the form (1.1) and then belong to the equation class discussed in this paper. So the first thing we needed to know was how many of the remaining $552 - 372 = 180$ examples admit linear symmetries of the form (1.1), and whether there was any other classification known for these 180 examples. The information for answering these and related questions is summarized in this table:

Class	First degree in y': 88	Higher degree in y': 92	Total: 180 equations
$[\xi = F,\ \eta = Py + Q]$	20	37	57
Abel (non-constant invariant)	15	0	15
Clairaut	0	15	15
d'Alembert	2	21	23
Unknown	33	21	54

Table 1. Classification of 180 *non* "linear, separable, Bernoulli, Riccati or Abel c.i." Kamke's examples.

Hence, $372 + 57 = 429$ equations out of Kamke's 552 solvable examples have linear symmetries of the form (1.1). Also, Table 1. shows that in Kamke, even among these particular 180 equations which exclude the "easy ones", there are more examples having symmetries of the form (1.1) than examples of Abel (non-constant invariant), Clairaut and d'Alembert types all together.[12]

In the second place, if we discard the 61 examples of Riccati-type found in Kamke, the solvable set is reduced to $552 - 61 = 491$ equations, and from this set, $429 - 61 = 368$ can be solved algorithmically as shown in sec. 2.

Moreover, a classification of Kamke's examples of Riccati type according to sec. 3 shows:

Class	$[\xi = F,\ \eta = Q]$	$[\xi = F,\ \eta = Py]$	"Two of $\{f, p, q\}$ are equal"	Total of equations
Riccati	7/61	22/61	2/61	31/61

Table 2. Classification according to sec. 3 of the 61 Riccati equations of Kamke's book.

So one half of these Riccati examples are still solvable using the algorithms described in sec. 3, without either mapping the problem into a second-order linear equation or having to solve auxiliary differential equations.

[10] The numbers of the Kamke examples we excluded in this way are: 47, 48, 50, 55, 56, 74, 79, 82, 202, 205, 206, 219, 234, 235, 237, 265, 250, 253, 269, 331, 370, 461, 503 and 576.

[11] For an enumeration of Kamke's examples of Abel type with constant invariant see [6] and concerning other classes see [4].

[12] We note there is no intersection between these classifications: all of Abel (non-constant invariant) Clairaut and d'Alembert equations do not have symmetries of the form (1.1).

5 Conclusions

In this work we have presented an algorithm for solving first-order equations, consisting of determining symmetries of the form (1.1). From the discussions of sec. 2, with these symmetries one can associate an equation class, represented by (2.4), which embraces all first-order equations mappable into separable ones through linear transformations. From the numbers of sec. 4, this class appears as the widest *first-order equation class* of which we are aware, all of whose members are algorithmically solvable as shown in sec. 2 or sec. 3, or mappable into second-order linear equations when the equation is of Riccati type but the methods in sec. 3 don't cover the case.

The algorithms presented neither require solving additional differential equations nor do they rely on the equation member of the class being *algebraic* (i.e.: rational in y and its derivatives) or on restrictions to the function fields; the only requirement on the functions entering the infinitesimals (1.1) are those implied by the fact that these infinitesimals do generate a Lie group of transformations.

Concerning other related works of which we are aware, the method presented in [15] for solving Abel equations with *constant* invariant is a particular case of the one presented here in that those equations are the subclass of (2.4) of Abel type. In the same direction, the method by Chini [7], a generalization of the method for Abel equations with constant invariant which solves more general equations of the form (1.4), is also a particular case in that (1.4) is a very restricted subclass of (2.4). Also, the class solvable through Chini's method is equivalent to a separable equation only through transformations (1.2) with $q = 0$. In this sense, the algorithm presented in sec. 2 generalizes both the one discussed in [15] and the one presented in [7].

The fact that this class (2.4) is algorithmically solvable makes this classification relevant for modern computer algebra implementations. As shown in sec. 4, taking as framework for instance Kamke's examples, 78% belong to this equation class. Even after discarding Riccati equations, 75% of the remaining Kamke examples belong to this class (2.4) and so are solvable by the algorithm presented in sec. 2. This algorithm actually solves many equation families not solved in the presently-available computer algebra systems (CAS). For instance, of the 4 examples shown in sec. 2.2, three cannot be solved by Maple 6 or Mathematica 4 at the time of writing this paper. Concerning the Riccati families presented in sec. 3, two of them (cases "$f = q$" and "$p = q$") are also not solved by these two CAS, which base their strategy in mapping Riccati equations into linear second-order ones.

An equally important differential equation problem, complementary to the one discussed in this paper, is the one where the equation (1.5) cannot be transformed into separable by means of linear transformations but it is still polynomial in the unknown. This problem was discussed at the end of the nineteenth century, first by Liouville then by Appel [12, 1], in the framework of classical invariant theory. The simplest version of this problem consists perhaps of Abel equations with non-constant invariant, for which a single class generalizing the known integrable classes is presented in the subsequent paper in this issue[17].

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References

[1] Appell P., 1889 Sur les invariants de quelques équations différentielles. Journal de Mathématique 5, 361-423.
[2] Bluman G. & Kumei S., 1996 *Symmetries and Differential Equations*. New York. Springer-Verlag.
[3] von Bülow, K., 2000 Equivalence methods for second order linear differential equations. M.Sc. Thesis, Faculty of Mathematics, University of Waterloo.
[4] Cheb-Terrab E.S., Duarte L.G.S. and da Mota L.A.C.P., 1997 Computer Algebra Solving of First Order ODEs Using Symmetry Methods. Computer Physics Communications, 101, 254.
[5] Cheb-Terrab E.S, Roche A.D., 1998 Symmetries and First Order ODE patterns. Computer Physics Communications 113, 239.
[6] Cheb-Terrab E.S, Roche A.D., Abel ODEs: Equivalence and New Integrable Cases. Computer Physics Communications 130, 2000, p.197.
[7] Chini M., 1924 Sull'integrazione di alcune equazioni differenziali del primo ordine. Rendiconti Instituto Lombardo (2) **57**, 506-511.
[8] Dickson L.E., 1924 Annals of Math. 2, 25, 324.
[9] Hereman W., 1995 *Chapter 13 in vol 3 of the CRC Handbook of Lie Group Analysis of Differential Equations*. Florida. Ed.: N.H.Ibragimov, CRC Press, Boca Raton.
[10] Hydon P.E., 1995 Conformal symmetries of first-order ordinary differential equations. J.Phys. A: Math. Gen. **27** 385-392.
[11] Kamke, E., 1947 *Differentialgleichungen*. N.Y. Chelsea Publ. Co.
[12] Liouville R., 1887 Sur une classe d'équations différentielles du premier ordre et sur les formations invariantes qui s'y rapportent. Comptes Rendus 103, 460-463.
[13] Murphy G.M., 1960 *Ordinary Differential Equations and their solutions*. Van Nostrand, Princeton.
[14] Pirondini G., 1904 Annali di Mat. 3 (9), 185.
[15] Schwarz F., 1998 Symmetry analysis of Abel's equation. Stud. Appl. Math. 100, no. 3, 269–294.
[16] Stephani H., 1989 *Differential equations: their solution using symmetries*. Cambridge University Press.
[17] Cheb-Terrab E.S, Roche A.D., An Abel ordinary differential equation class generalizing known integrable classes. European Journal of Applied Mathematics - this issue.

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