Combinatorics and PDE's

Q1: Let $a(n)$ be the number of solutions to $\pm 1 \pm 2 \cdots \pm n = 0$

Estimate $a(n)$ for large n.

Q2: Let $b(m,n)$ be the number of subsets of $\{1, 2, \ldots n\}$ whose elements add up to m.

Estimate $b(m,n)$ for large n and m.

For Q1, $a(n)$ is the constant term in the expansion of

$$(1) \qquad (z^{-1}+z^{+1})(z^{-2}+z^{2}) \cdots (z^{-n}+z^{n}).$$

Ex: $(z^{-1}+z^{1})(z^{-2}+z^{2})(z^{-3}+z^{3}) = z^{-6} + z^{-4} + z^{-2} + \boxed{2} z^{0} + z^{2} + z^{4} + z^{6}$

$-1-2+3$
$+1+2-3$

$\Rightarrow a(3) = 2$

Similarly for Q2, $b(m,n)$ is the coefficient of x^m in the expansion of

$$(2) \qquad (1+x)(1+x^2) \cdots (1+x^n)$$

Here is a table of $b(m,n)$:

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	1																					
1	1	1																				
2	1	1	1	1																		
4	1	1	1	2	1	1	1															
5	1	1	1	2	2	2	2	2	1	1	1											
6	1	1	1	2	2	3	3	3	3	3	3	2	2	1	1	1						
7	1	1	1	2	2	3	4	4	4	5	5	5	4	4	4	3	2	2	1	1	1	

Ex: $6 = 1+2+3 = 1+5 = 2+4 \Rightarrow \begin{cases} b(6,5) = 3 \\ b(6,4) = 2 \end{cases}$

We have:
$$\begin{cases} b(m,n) = b(m-n, n-1) + b(m, n-1) \\ b(m,0) = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases} \end{cases} \tag{3}$$

In fact this recursion defines $b(m,n)$.

Motivation: Numerical experiments suggest that $m \to b(m,n)$ looks like a Gaussian for large n (see next page). From (3), we see that $\sum_{m=-\infty}^{\infty} b(m,n) = 2^n$

So rescale, let $b(m,n) = 2^n u(m,n)$.

then $u(m,n) = \frac{1}{2}\left(u(m-n, n-1) + u(m, n-1)\right)$.

Next note that $m \in [0, \frac{n(n+1)}{2}]$.

Now fix $N >> 1$ and let $0 \leq n \leq N$.

Then $m \in [0, \frac{N(N+1)}{2}] \sim [0, \frac{N^2}{2}]$

So to pass to the continuous limit, we let $x = \frac{m}{N^2}$, $t = \frac{n}{N}$

and let $h = \frac{1}{N}$; then $t \in [0,1]$; $x \in [0, \frac{1}{2}]$

and we define: $V(x,t) = u(m,n) =$

Then

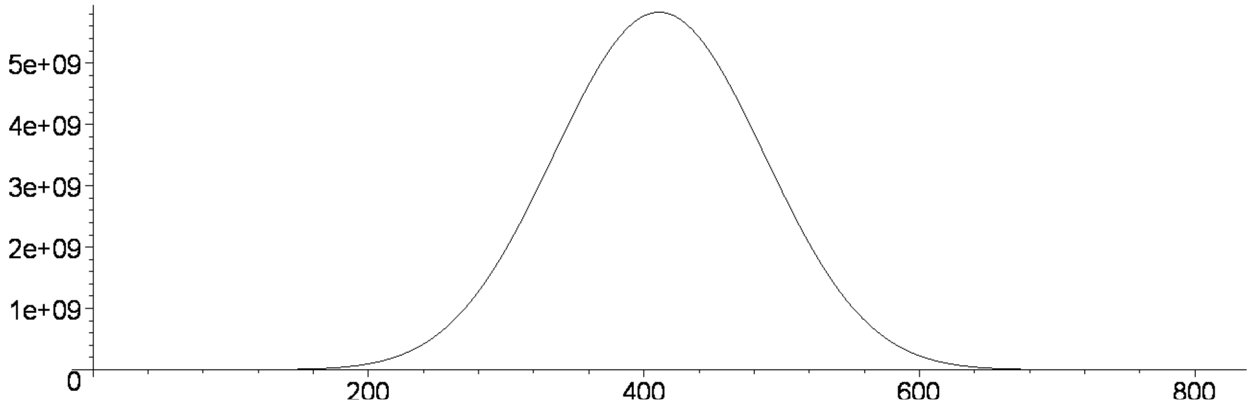
$$u(m,n) = V\left(\frac{m}{N^2}, \frac{n}{N}\right)$$
$$u(m, n-1) = V\left(\frac{m}{N^2}, \frac{n-1}{N}\right) = V(x, t-h)$$
$$u(m-n, n-1) = V\left(\frac{m-n}{N^2}, \frac{n-1}{N}\right) = V(x - ht, t-h)$$

```
>
N := 40;
p := expand(product(1+x^(n), n=1..N)):
L:=convert(PolynomialTools[CoefficientVector](p, x), list):
mmax := max(op(map(abs, L)));
```

$$N := 40$$

$$mmax := 5830034720$$

```
> PLOT(CURVES([seq([i, L[i]], i=1..nops(L))]));
```

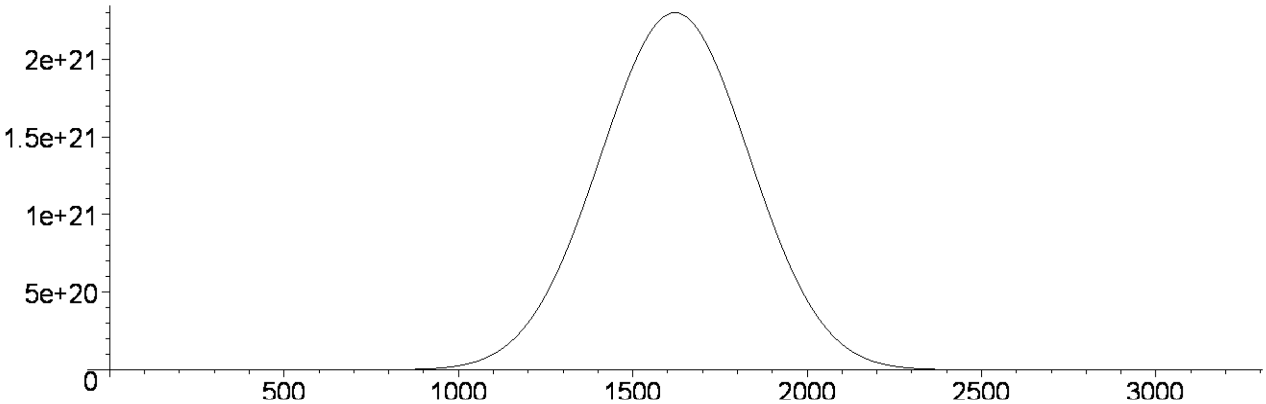



```
>
N := 80;
p := expand(product(1+x^(n), n=1..N)):
L:=convert(PolynomialTools[CoefficientVector](p, x), list):
mmax := max(op(map(abs, L)));
```

$$N := 80$$

$$mmax := 2300241216389780443900$$

```
> PLOT(CURVES([seq([i, L[i]], i=1..nops(L))]));
```

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>
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So we get: $\quad v(x,t) = \frac{1}{2}\left[v(x-ht, t-h) + v(x, t-h)\right]$

Expand in Taylor series for small h:

(4)' $\cancel{v(x,t)} \approx \cancel{v(x,t)} - \frac{ht}{2} v_x - h v_t + \frac{1}{4}(ht)^2 v_{xx} \cdots$

At leading order, we get $\underline{v_t} \sim -\underline{\frac{t v_x}{2}} + h\frac{t^2}{4} v_{xx} \cdots$

The v_x term corresponds to the drift.

This suggests further change of variables,

let $\boxed{y = x - \frac{t^2}{4}}$ so that

$$v(x,t) = w(y,t) = w\left(x - \tfrac{t^2}{4}, t\right)$$

Then $v(x, t-h) = w\left(x - \frac{(t-h)^2}{4}, t-h\right) = w\left(x - \frac{t^2}{4} + \frac{ht}{2}, t-h\right)$

$$= w\left(y + \tfrac{ht}{2}, t-h\right)$$

$$v(x-ht, t-h) = w\left(y - \tfrac{ht}{2}, t-h\right)$$

(5) $\Rightarrow \quad w(y,t) = \frac{1}{2}\left[w\left(y - \frac{ht}{2}, t-h\right) + w\left(y + \frac{ht}{2}, t-h\right)\right]$

Taylor expansion of (5) then yields:

(6) $\boxed{w_t = \frac{ht^2}{8} w_{yy} + O(h^2)}$

Warning: It is incorrect to directly substitute $y = x - \frac{t^2}{4}$ into (4); this would give $w_t = h\frac{t^2}{4} w_{yy}$ which is wrong const.

Next we note that $\sum_m u(m,n) = 1 \Rightarrow \int_{-\infty}^{\infty} w(y,t)\,dt = h^2$

and $\underset{(= V(X,0))}{w(y,0)} = \begin{cases} 0, & \text{if } y > h^2 \\ 1, & \text{if } y = 0 \end{cases}$

So we approximate $w(y,0) = h^2\, \delta(y)$

where $\delta(y)$ is the delta function.

Now (6) is a rescaled heat equation:

let $t = f(s)$ then $\dfrac{w_s}{f'(s)} = h\dfrac{(f(s))^2}{8}\, w_{yy}$

So choose $f'(s) f^2 \dfrac{h}{8} = 1 \Rightarrow \dfrac{f^3}{3}\dfrac{h}{8} = s$

i.e. $\boxed{s = \dfrac{h}{24} t^3}$ then $\begin{cases} w_s = w_{yy} \\ w(y,0) = h^2\,\delta(y) \end{cases}$ (7)

The eq'n (7) has solution given by the heat kernel:

$$w(y,s) = h^2 \frac{1}{\sqrt{4\pi s}}\, e^{-\frac{y^2}{4s}} \quad ;$$

going back to $b(m,n)$ this gives:

$$v(x,t) = \sqrt{\frac{6}{\pi}} \left(\frac{h}{t}\right)^{\frac{3}{2}} e^{-\frac{6\left(x - \frac{t^2}{4}\right)^2}{h t^3}}, \quad h \to 0$$

(8) $$b(m,n) = 2^n \sqrt{\frac{6}{\pi}}\, n^{-3/2}\, e^{-\frac{6}{n}\left[\frac{m}{n} - \frac{n}{4}\right]^2}, \quad \underset{n\to\infty}{m - \frac{n^2}{4} \leq O(n)}$$

(9) $$\boxed{b\left(\frac{n(n+1)}{4}, n\right) \sim \sqrt{\frac{6}{\pi}}\, 2^n\, n^{-3/2}, \quad n \to \infty}$$

Remarks:

- The formula (8) is <u>not</u> valid when $m \ll n^2$.
  For example, note that $b(m,n) = b(m,\infty)$ if $n \geq m$.
  [i.e. the coefficient of x^m in $(1+x)(1+x^2)(1+x^3)\cdots$]
  In fact, $b(m,\infty)$ is the partition function "Q", its asymptotics are:

$$b(m,\infty) \sim \frac{e^{\pi\sqrt{n/3}}}{4\cdot 3^{1/4}\, n^{3/4}}$$

- Error in (9) is of $O(\frac{1}{n})$ as the following table illustrates:

n	$b(\frac{n(n+1)}{4}, n)$ [exact]	$b(\frac{n(n+1)}{4}, n)$ [using (9)]	Error
20	15272	16201.5	5.74%
40	5.830034720	6.006×10^{10}	2.94%
80	2.30024×10^{22}	2.335×10^{22}	1.48%

Next, note that

$$(z^{-1}+z^{+1})\cdots(z^{-n}+z^{n}) = (1+z^2)(1+z^4)\cdots(1+z^{2n})\, z^{-\frac{(n(n+1))}{2}}$$

$$= P_n(z^2)\, z^{-\frac{(n(n+1))}{2}}$$

where $P_n(X) = (1+X)\cdots(1+X^n)$.

Thus $a(n) = \begin{cases} b\left(\frac{n(n+1)}{4}, n\right) & \text{if } n \text{ or } n+1 \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$

This gives the answer to Q1:

$$a(n) \sim \begin{cases} \sqrt{\frac{6}{\pi}}\, 2^n\, n^{-3/2}, & n \to \infty \text{ with } n \sim 3, 4, 7, 8, 13, 14, \ldots \text{ (i.e. } n \equiv 3 \text{ or } 0 \pmod 4) \\ 0 & \text{otherwise.} \end{cases}$$

Open Questions:

Let $c(m,n) \equiv$ number of ways to write m as a sum of integers from 1 to n

Ex: $c(5,3)$:

$$
\begin{aligned}
5 &= 1+1+1+1+1 \\
&= 1+1+1+2 \\
&= 1+1+3 \\
&= \cancel{1+4} \\
&= 1+2+2 \\
&= 2+3
\end{aligned}
$$

$c(5,3) = 5$

Then $c(m,n) \equiv$ coeff of x^m in the expansion of

$$(1+x+x^2 \cdots)(1+x^2+x^4+x^6\cdots)\cdots(1+x^n+x^{2n}+\cdots)$$

$$= \frac{1}{(1-x)(1-x^2)\cdots(1-x^n)}$$

If $n \geq m$ then $c(m,n) = c(m,\infty)$ was found by Ramanudjan-Hardy:

$$c(m,\infty) \sim \frac{1}{4\sqrt{3}\,n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

But what if $m \gg n \gg 1$?

Exercise: Let $b_2(m,n)$ be the number of subsets of $\{1, 4, 9, \dots n^2\}$ whose elements add up to m.

Show that

$$\boxed{b_2\left(\frac{n^3}{6}, n\right) \sim \sqrt{\frac{10}{\pi}}\, n^{-5/2}\, 2^n \quad \text{as } n \to \infty}$$

Exercise: Let $c(n)$ be the number of solutions to

$$0 = a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \dots + a_n\, n$$

where a_i is one of $+1, -1$ or 0.

Show that

$$c(n) \sim \frac{3}{2\sqrt{\pi}}\, n^{-3/2}\, 3^n$$

[See S. Finch, http://algo.inria.fr/csolve/signum.pdf]