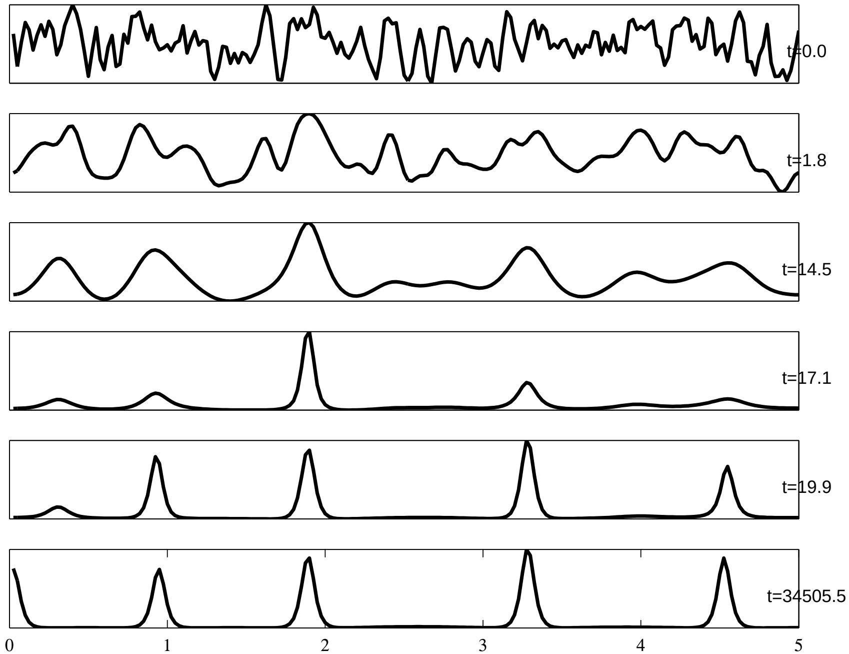
Hot spots in crime model



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Joint works with Jonah Breslau, Tum Chaturapruek, Daniel Yazdi, Scott McCalla, Michael Ward, Juncheng Wei.

UCLA Model of hot-spots in crime

- Originally proposed by Short, D'Orsogna, Pasour, Tita, Brantingham, Bertozzi, and Chayes, 2008 [The UCLA model]

- Crime is ubiquious but not uniformly distributed
  - Some neigbourhoods are worse than others, leading to crime ”hot spots”
  - Crime hotspots can persist for long time.

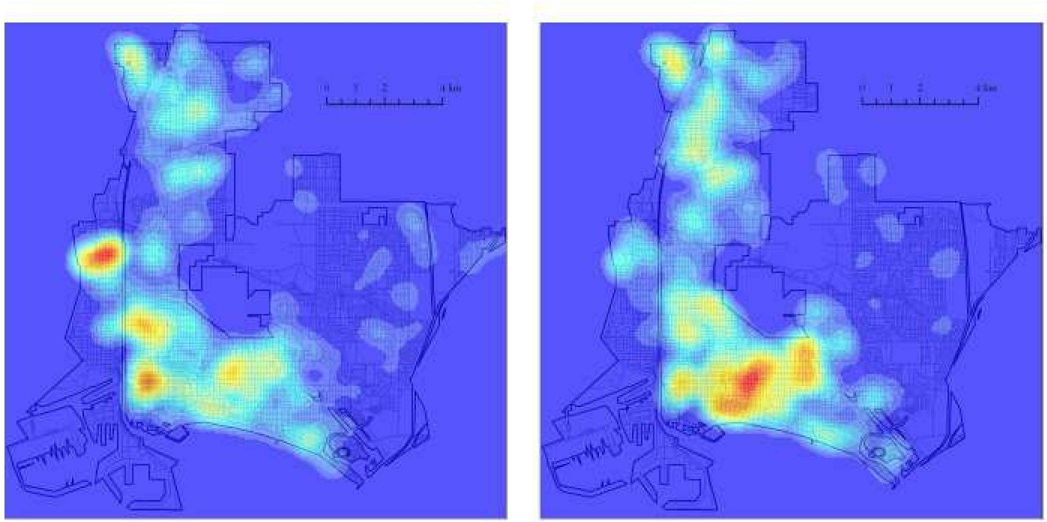


Fig. 1. Dynamic changes in residential burglary hotspots for two consecutive three-month periods beginning June 2001 in Long Beach, CA. These density maps were created using ArcGIS.

Figure taken from Short et.al., ***A statistical model of criminal behaviour,*** 2008.

- Crime is temporaly correlated:
  - Criminals often return to the spot of previous crime
  - If a home was broken into in the past, the likelyhood of subsequent breakin increases
  - Example: graffitti ”tagging”

Modelling criminal's movement

- In the original model, biased Brownian motion was used to model criminal's movement

- Our goal is to extend this model to incorporate more realistic motion

- Typical human motion consists short periods of fast movement [car trips] interspersed with long periods of slow motion [pacing, thinking about theorems, sleeping...]

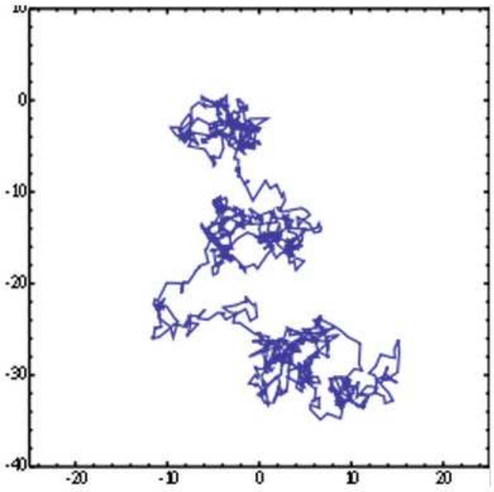
- Such motion is often modelled using **Levi Flights:** At each time, the speed is chosen according to a **power-law distribution**; direction chosen at random: $|y(t+\delta t)-y(t)| = \delta t X$ where X is a power-law distribution whose distribution function is

$$f(d) = C\,|d|^{-\mu}$$

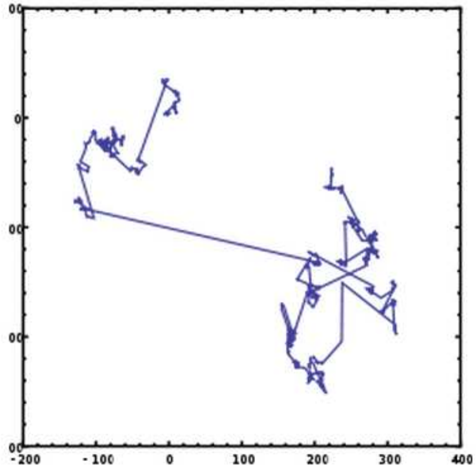
- μ is the power law exponent

  - In 1D, $1 < \mu \leq 3$; in 2D, $1 < \mu \leq 4$.

  - $\mu = 3$ corresponds to Brownian motion in one dimension.



Brownian motion



Levi flight motion

- González, Hidalgo, Barabási, *Understanding individual human mobility patterns, Nature 2008,* use cellphone data to suggest that human motion follows “truncated” Levi flight distribution with $\mu \approx 2.75$.

Discrete (cellular automata) model

- Two variables

$$A_k(t) \equiv \text{attractiveness at node } k, \text{ time } t;$$
$$N_k(t) \equiv \text{criminal density at node } k$$

- **Modelling attractiveness:** Attractiveness has static and dynamic component:

$$A_k(t) \equiv A^0 + B_k(t).$$

$$B_k(t+\delta t) = \underbrace{\left[(1-\hat{\eta})\,B_k(t) + \frac{\hat{\eta}}{2}\,(B_{k-1}+B_{k+1})\right]}_{\text{"broken window effect"}} \underbrace{(1-w\delta t)}_{\text{decay rate}} + \underbrace{\delta t A_k N_k \theta}_{\text{\# of robberies}}.$$

- $0 < \hat{\eta} < 1$ is the strength of broken window effect

- w is the decay rate

- **Modelling criminal movement:** Define the ***relative weight*** of a criminal moving from node i to node k, where $i \neq k$, as

$$w_{i \to k} = \frac{A_k}{l^\mu \left| i - k \right|^\mu}. \tag{1}$$

  - l is the grid spacing, μ the Levi flight power law exponent
  - The weight is ***biased*** by attractiveness field

- The ***transition probability*** of a criminal moving from point i to point k, where $i \neq k$, is

$$q_{i \to k} = \frac{w_{i \to k}}{\sum_{j \in \mathbb{Z}, j \neq i} w_{i \to j}}. \tag{2}$$

- Update rule for criminal density:

$$N_k(t + \delta t) = \sum_{i \in \mathbb{Z}, i \neq k} N_i \cdot (1 - A_i \delta t) \cdot q_{i \to k} + \Gamma \delta t. \tag{3}$$

  - $A_i \delta t \equiv$ probability that criminal robs
  - $(1 - A_i \delta t) \equiv$ probability that no robbery occurs
  - $N_i \cdot (1 - A_i \delta t) \equiv$ expected number of criminals at node i that don't rob
  - $N_i \cdot (1 - A_i \delta t) \cdot q_{i \to k} \equiv$ expected number of criminals that move from mode i to mode k.
  - $\Gamma \delta t \equiv$ constant "feed rate" of the criminals

Take a limit $l, \delta t \ll 1$:

- Main trick is to write $A_i \sim A(x)$ where $x = li$; then

$$\begin{aligned}\sum_{j\in\mathbb{Z}, j\neq i} w_{i\to j} &= \sum_{j\in\mathbb{Z}, j\neq i} \frac{A_j}{l^\mu |i-j|^\mu} \\ &= \sum_{j\in\mathbb{Z}, j\neq i} \frac{A_j - A_i}{l^\mu |i-j|^\mu} + \sum_{j\in\mathbb{Z}, j\neq i} \frac{A_i}{l^\mu |i-j|^\mu} \\ &\sim \frac{1}{l}\int_{-\infty}^{\infty} \frac{A(y)-A(x)}{|x-y|^\mu} dy + l^{-\mu} 2\zeta(\mu) A(x)\end{aligned} \tag{4}$$

- We recognize the integral as ***fractional Laplacian,***

$$\Delta^s f(x) = 2^{2s} \frac{\Gamma(s+1/2)}{\pi^{1/2}|\Gamma(-s)|} \int_{-\infty}^{\infty} \frac{f(x)-f(y)}{|x-y|^{2s+1}} dy, \quad 0 < s \leq 1.$$

- Key properties:
  - The normalization constant is chosen so that the Fourier transform is:

$$\mathcal{F}_{x\mapsto q}\{\Delta^s f(x)\} = -|q|^{2s} \mathcal{F}_{x\mapsto q}\{f(x)\}. \tag{5}$$

  - $s = 1$ corresponds to the usual Laplacian: $\Delta^s f(x) = f_{xx}$ if $s = 1$.

Continuum model

The continuum limit of CA model becomes

$$\frac{\partial A}{\partial t} = \eta A_{xx} - A + \alpha + A\rho. \tag{6}$$

$$\frac{\partial \rho}{\partial t} = D\left[A\Delta^s\left(\frac{\rho}{A}\right) - \frac{\rho}{A}\Delta^s(A)\right] - A\rho + \beta \tag{7}$$

where

$$s = \frac{\mu - 1}{2} \in (0,1]; \quad \eta = \frac{l^2\hat{\eta}}{2\delta t w}; \quad D = \frac{l^{2s}}{\delta t}\frac{\pi^{1/2}2^{-2s}|\Gamma(-s)|}{z\Gamma(2s+1)w}; \quad \alpha = A_0/w; \;\; \beta = \Gamma\theta/w^2.$$

- Separation of scales: if $l, \delta t \ll 1$ then

$$D\eta^{-s} \gg 1; \quad 0 < s \leq 1. \tag{8}$$

- The special case $s = 1$ ($\mu = 3$) corresponds to regular diffusion $\Delta^1 f(x) = f_{xx}$.
  - We recover the UCLA model because:

$$A\left(\frac{\rho}{A}\right)_{xx} - \frac{\rho}{A}A_{xx} = \left(\rho_x - 2\frac{\rho}{A}A_x\right)_x$$

  - Note that $D \to \infty$ as $s \to 1^-$ since $|\Gamma(-s)| \sim 1/(1-s)$.

Simulation of continuum model

- Use a spectral method in space combined with method of lines in time.

- That is, we first discretize in space $x \in [0, L]$. To approximate $\Delta^s u$, we make use of Fourier transform:
$$\Delta^s u = \mathcal{F}^{-1}\left(-|q|^{2s}\mathcal{F}_{x\mapsto q}\{u\}\right). \tag{9}$$

- This becomes FFT on a bounded interval

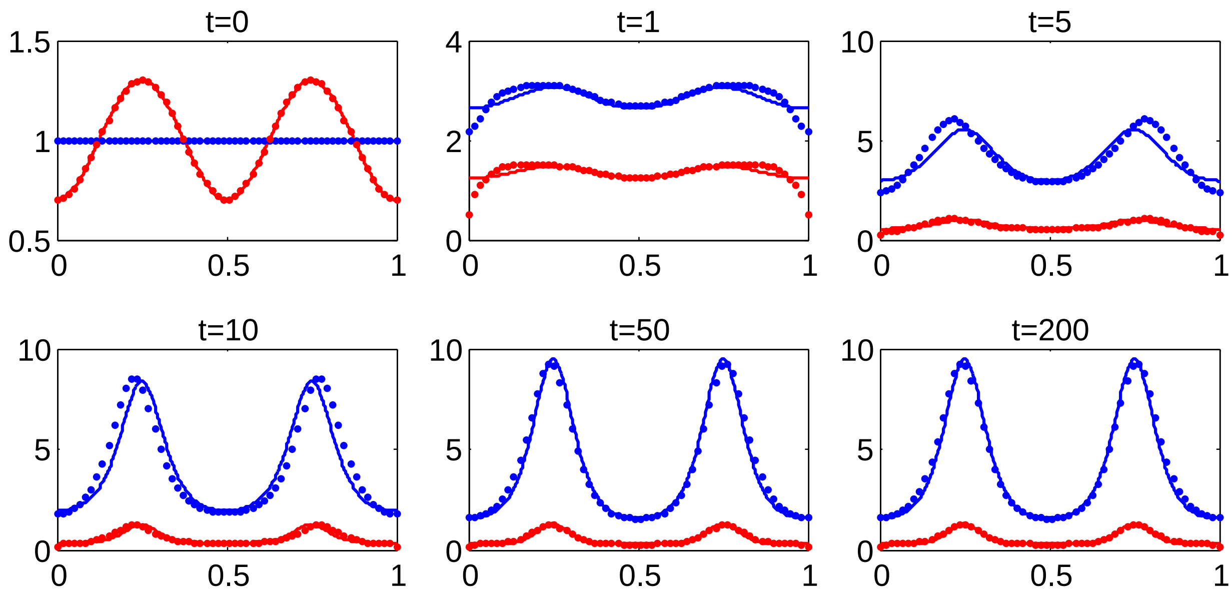
- ***Matlab code*** to estimate the discretization of $\Delta^s u(x), \quad x \in [0, 1]$:
```
n = numel(u);
q = 2*pi*[0:n/2-1, -n/2:-1]';
LaplaceS_u = ifft(-q.^(2*s).*fft(u));
```

- This implicitly imposes periodic boundary conditions on the solution.

Comparison: discrete vs. continuum

Example: Take $\mu = 2.5$, $n = 60$, $l = 1/60$, $\hat{\eta} = 0.1$, $\delta t = 0.01$, $A_0 = 1$, $\Gamma = 3$.

Then the continuum model gives $s = 0.75$, $\eta = 0.001388$, $D = 0.1828$, $\alpha = 1$, $\beta = 3$.



Discrete model is represented by dots; continuum model by solid curves. Blue is A, red is ρ. Two hot-spots form.

Turing instability analysis

$$\frac{\partial A}{\partial t} = \eta A_{xx} - A + \alpha + A\rho, \qquad \frac{\partial \rho}{\partial t} = D\left[A\Delta^s\left(\frac{\rho}{A}\right) - \frac{\rho}{A}\Delta^s(A)\right] - A\rho + \beta$$

Steady state:

$$\bar{A} = \alpha + \beta; \quad \bar{\rho} = \frac{\beta}{\alpha + \beta}.$$

Linearization:

$$A(x,t) = \overline{A} + \phi e^{\lambda t} e^{ikx}, \tag{10a}$$

$$\rho(x,t) = \overline{\rho} + \psi e^{\lambda t} e^{ikx}. \tag{10b}$$

Using the Fourier transform property, we have:

$$\Delta^s e^{ikx} = -|k|^{2s} e^{ikx}$$

so the eigenvalue problem becomes

$$\begin{bmatrix} -\eta|k|^2 - 1 + \bar{\rho} & \bar{A} \\ \frac{2\overline{\rho}}{\bar{A}} D|k|^{2s} - \bar{\rho} & -D|k|^{2s} - \bar{A} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \lambda \begin{bmatrix} \phi \\ \psi \end{bmatrix}. \tag{11}$$

The dispersion relationsh is then given by

$$\lambda^2 - \tau\lambda + \delta = 0$$

where

$$\tau = -D|k|^{2s} - \eta|k|^2 - \bar{A} - 1 + \bar{\rho}; \qquad \delta = D|k|^{2s}\left(\eta\,|k|^2 + 1 - 3\bar{\rho}\right) + \eta\,|k|^2\,\bar{A} + \bar{A}.$$

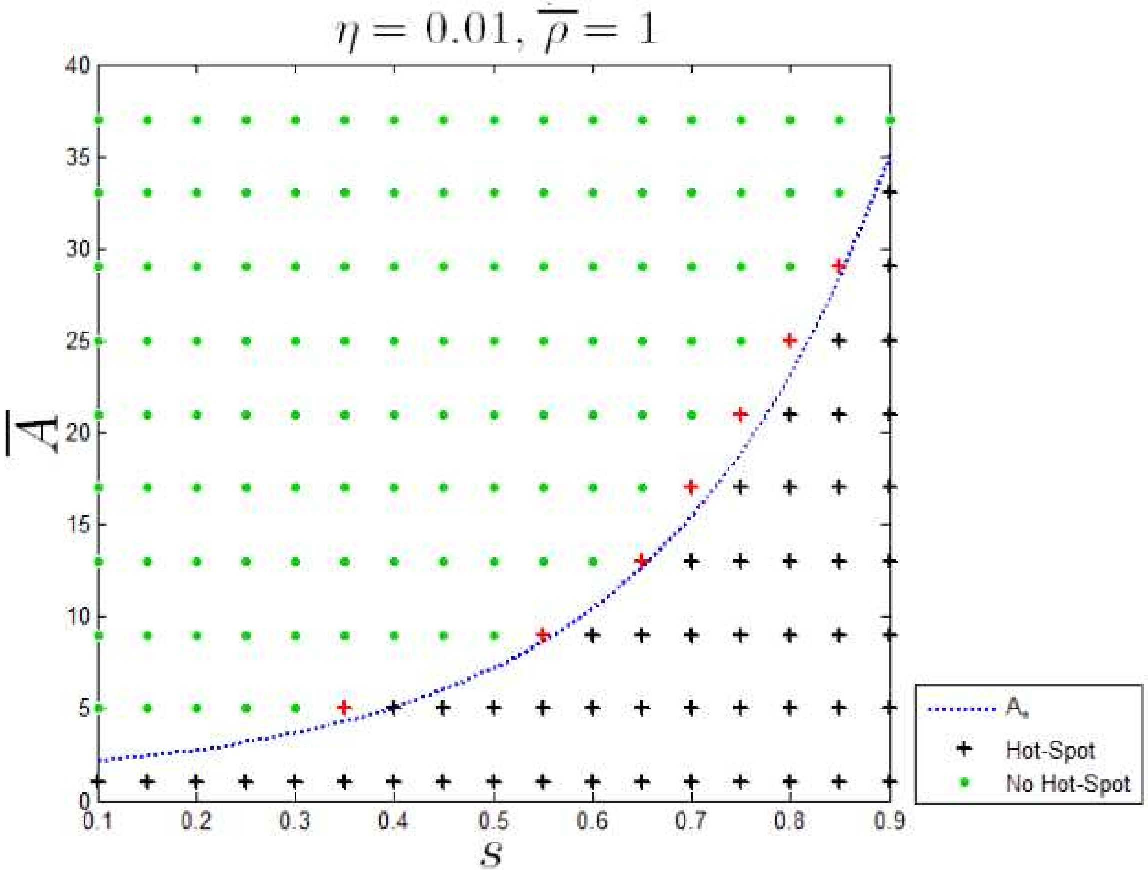
Note that $\tau < 0$ so the steady state is stable iff $\delta > 0$ ***for all*** k. Equilibrium is stable if $\bar{\rho} < 1/3$. If $\bar{\rho} > 1/3$ then equilibrium is unstable iff

$$\bar{A} < D\eta^s x^s \left(-1 + \frac{3\bar{\rho}}{x+1}\right) \tag{12}$$

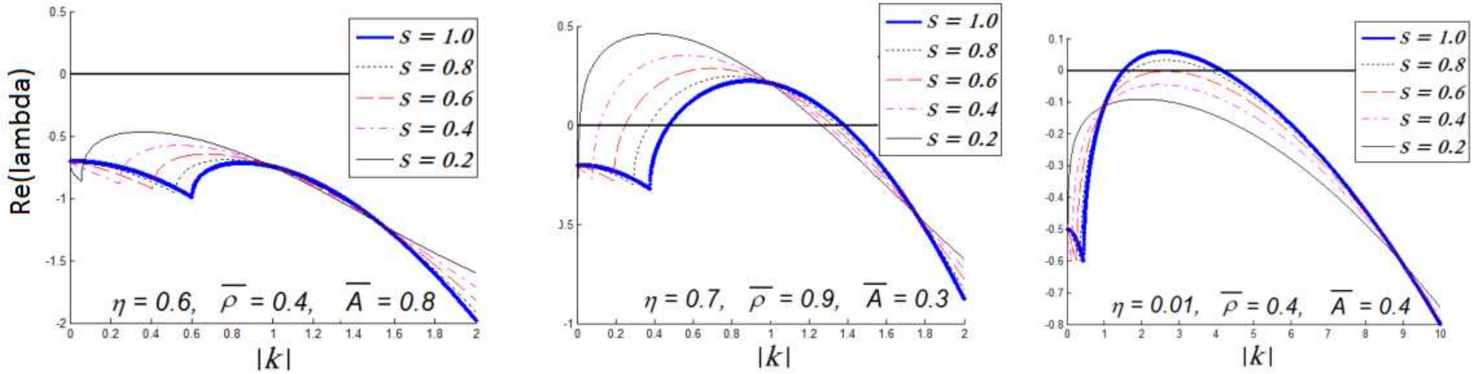
where x is the unique positive root of

$$x^2 + x\left(2 + 3\bar{\rho}(1-s)/s\right) + 1 - 3\bar{\rho} = 0.$$

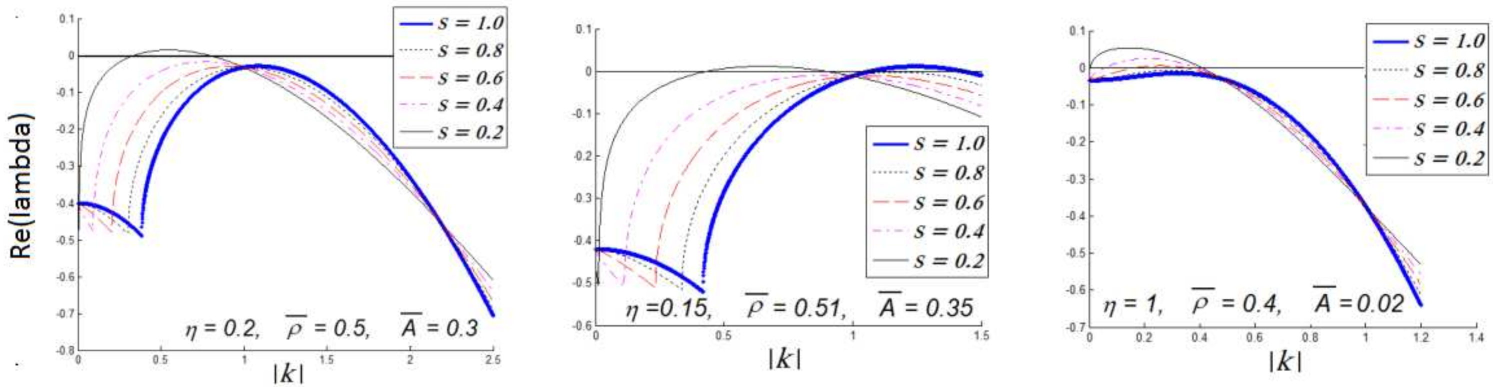
Comparison with numerics



The effect of changing s on dispersion relationship



(a) A stable regime (b) An unstable regime (c) Fractional diffusion leads to stability



(d) Fractional diffusion leads to unstability (e) Fractional diffusion leads to stability and then instability (f) A regime in which $|k_2| < |k_1| = 1$

Dominant instability [biggest λ]

- Recall that in terms of original gridsize l and time step δt, we have:

$$s = \frac{\mu - 1}{2} \in (0,1]; \quad \eta = \frac{l^2 \hat{\eta}}{2\delta t w}; \quad D = \frac{l^{2s}}{\delta t} \frac{\pi^{1/2} 2^{-2s} |\Gamma(-s)|}{z\Gamma(2s+1) w}$$

so that $\eta^{-s} D = O((1-s)^{-1} (\delta t)^{s-1}) \gg 1, \;\; 0 < s \leq 1$

- ***For a physically relevant regime, the continuum model satisfies the key relationship***

$$\boxed{\eta^{-s} D \gg 1.} \qquad (13)$$

Change the variables $k = x^{1/2}\eta^{-1/2}$ and let $M = D\eta^{-s} \gg 1$. Then we obtain

$$\tau = -Mx^s - x^2 + \bar{\rho} - 1 - \bar{A}; \;\; \delta = Mx^s(x + 1 - 3\bar{\rho}) + x\bar{A} + \bar{A}.$$

The fastest growing mode corresponds to the maximum of the dispersion curve:

$$\lambda^2 - \tau\lambda + \delta = 0 \quad \text{and} \quad \lambda = \tau_x / \delta_x.$$

- Asymptotically, this becomes

$$k_{\text{fastest}}(s) \sim \left[\frac{s\bar{\rho}(-2+3\bar{A}+6\bar{\rho})}{D\eta}\right]^{\frac{1}{2(s+1)}}, \quad D\eta^{-s} \gg 1. \tag{14}$$

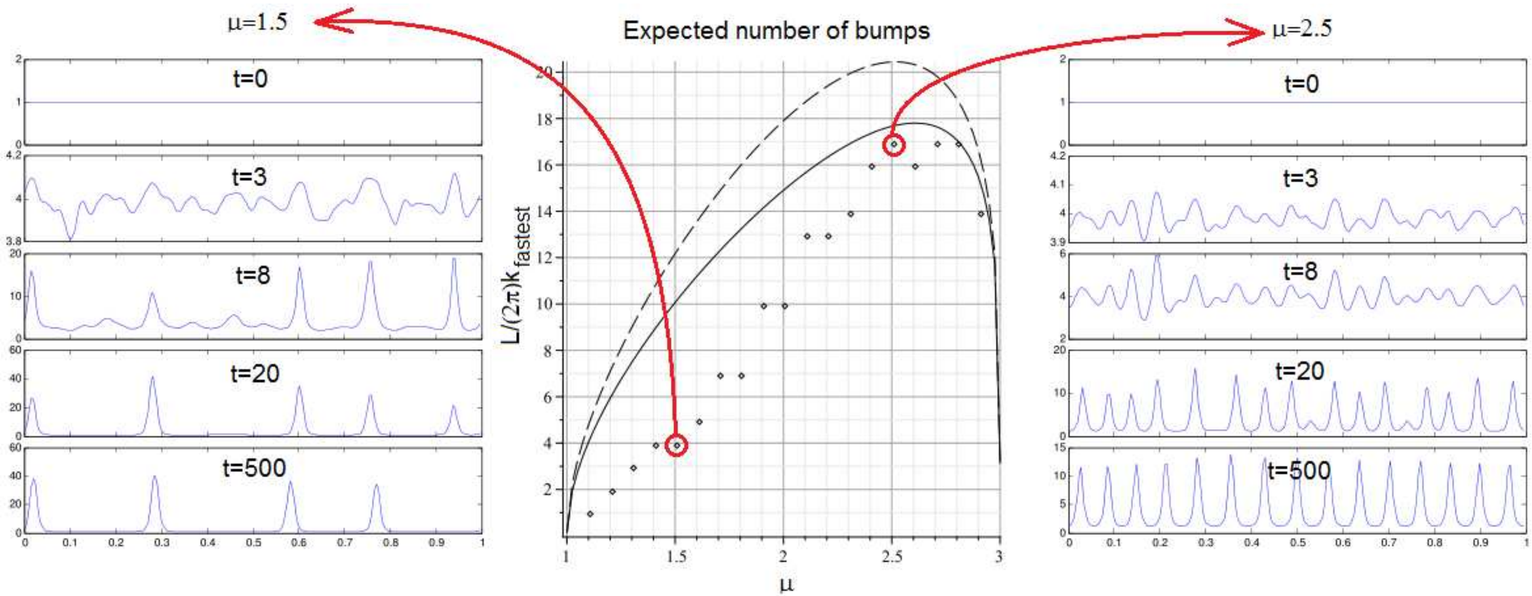
$$\text{Expected number of “bumps”} \approx \text{floor}\left(\frac{L}{2\pi}k_{\text{fastest}}\right). \tag{15}$$

- k_{fastest} is at a maximum when s satisfies

$$\log\left(\frac{\bar{\rho}(-2+3\bar{A}+6\bar{\rho})}{D\eta}s\right) = s+1$$

Comparison with numerics

$$l = 0.01, \delta t = 0.05, \hat{\eta} = 0.02, A_0 = 1, \Gamma = 3$$



- The initial instability has sinusoidal shape

- Eventually, hot-spot forms.
  - Hot-spots are localized regions which are ***not*** of the sinusoidal shape!
  - In general, the total number of stable hot-spots ***does not*** correspond to fastest-growing Turing mode!
  - The hot-spot regime is separate from the Turing regime!

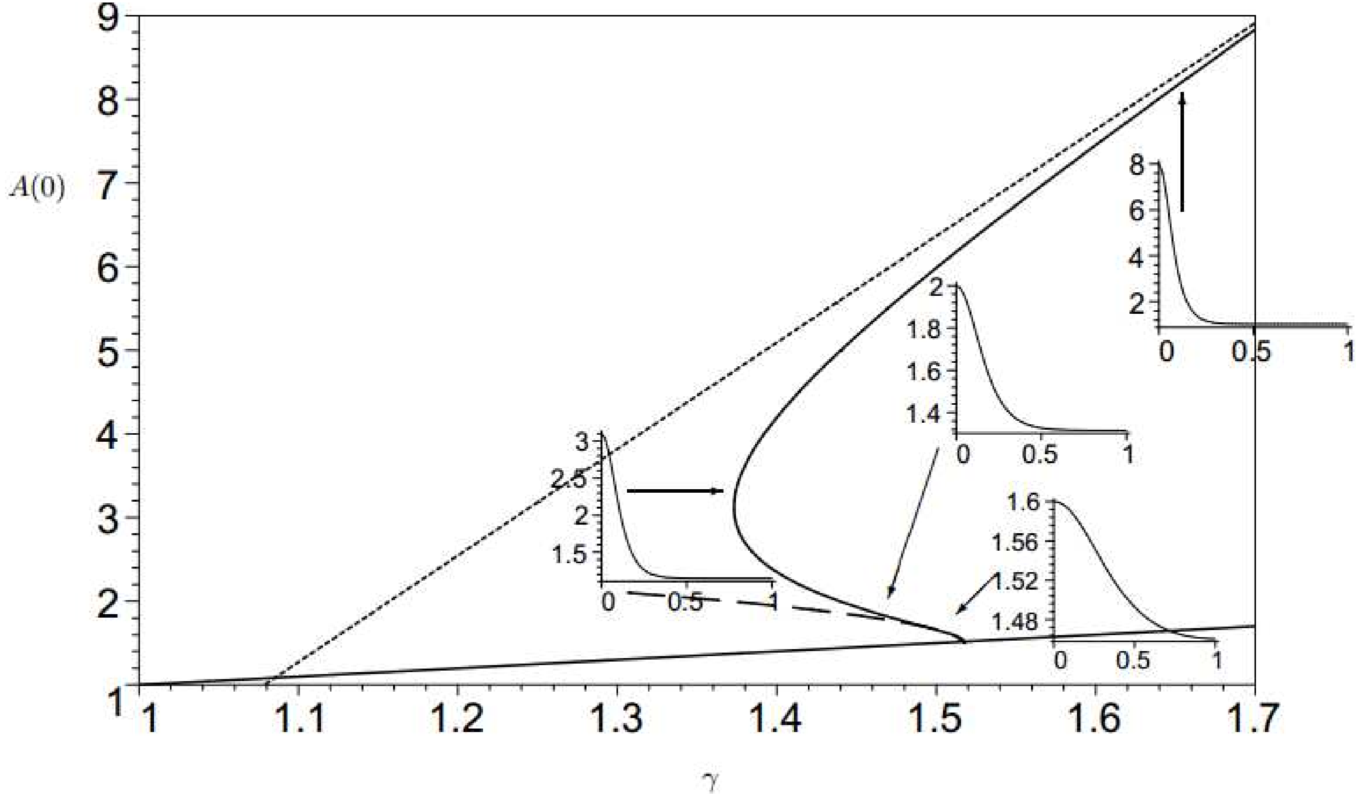


FIGURE 7. Numerically computed bifurcation diagram of $A(0)$ vs. γ. The parameter values are $\alpha = 1, \varepsilon = 0.05, x \in [0,1]$, and $D = 2$. A localized hot-spot appears for large values of $A(0)$. The asymptotics $A(0) \sim \frac{2(\gamma-\alpha)}{\varepsilon\pi}$ (see (2.19)) are shown by a dotted line. The constant steady state $A \sim \gamma$ is indicated by a solid straight line line. Turing patterns are born from the spatially uniform steady state as a result of a Turing bifurcation at $\gamma \sim 3\alpha/2 = 1.5$. The weakly nonlinear regime is indicated by a dashed parabola coming out of the bifurcation point. Inserts shows the change in the shape of the profile $A(x)$ along the bifurcation curve.

Construction of hotspot solution

Hotspot solution satisfies:

$$0 = \eta A_{xx} - A + \alpha + A\rho; \quad 0 = D\left[A\Delta^s\left(\frac{\rho}{A}\right) - \frac{\rho}{A}\Delta^s(A)\right] - A\rho + \beta \qquad (16)$$

and is periodic on $[-1, 1]$.

- **Key transformation:** Let $\rho = vA^2$; then

$$0 = \eta A_{xx} - A + \alpha + A^3 v; \quad 0 = D\left[A\Delta^s(vA) - vA\Delta^s(A)\right] - A^3 v + \beta \qquad (17)$$

- **Inner problem:** Change variables $x = \eta^{1/2} y$; then

$$0 = A_{yy} - A + \alpha + A^3 v; \quad 0 = D\eta^{-s}\left[A\Delta^s(vA) - vA\Delta^s(A)\right] - A^3 v + \beta$$

- As before, $D\eta^{-s} \gg 1$ so that in the inner region,

$$A\Delta_y^s(vA) - vA\Delta_y^s(A) \sim 0 \implies v(y) \sim \text{const.} \sim v_0$$

  - Change variables $A = v_0^{-1/2} w(y)$, then

$$w_{yy} - w + w^3 = 0 \qquad \Longrightarrow \qquad w = \sqrt{2}\,\text{sech}(y)$$

- To determine v_0, integrate (17) and use the identity $\int f\Delta^s g - g\Delta^s f = 0$; then

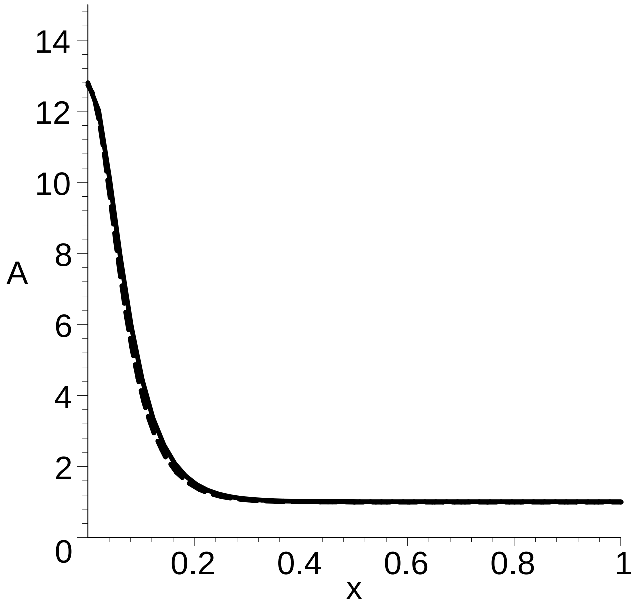
$$\int A^3 v_0 \sim \int \beta$$

- The final result is

$$A(x) \sim \begin{cases} A_{\max} w(x/\sqrt{\eta}), & x = O(\varepsilon) \\ \alpha, & x \gg O(\varepsilon). \end{cases}$$

$$A_{\max} \sim \frac{2l\beta\pi^{-3/2}}{\sqrt{\eta}}$$

where l is the half-width of the spot.



Stability of hot-spots (1D, $s = 1$)

- **Localized states:** Consider a periodic pattern consisting of ***localized*** hotspots of radius l. It is stable iff $l > l_c$ where

$$l_c := \frac{(\eta D)^{1/4}\,\pi^{1/2}\alpha^{1/2}}{\beta^{3/4}}.$$

- **Turing instability in the limit $\varepsilon \to 0$:**

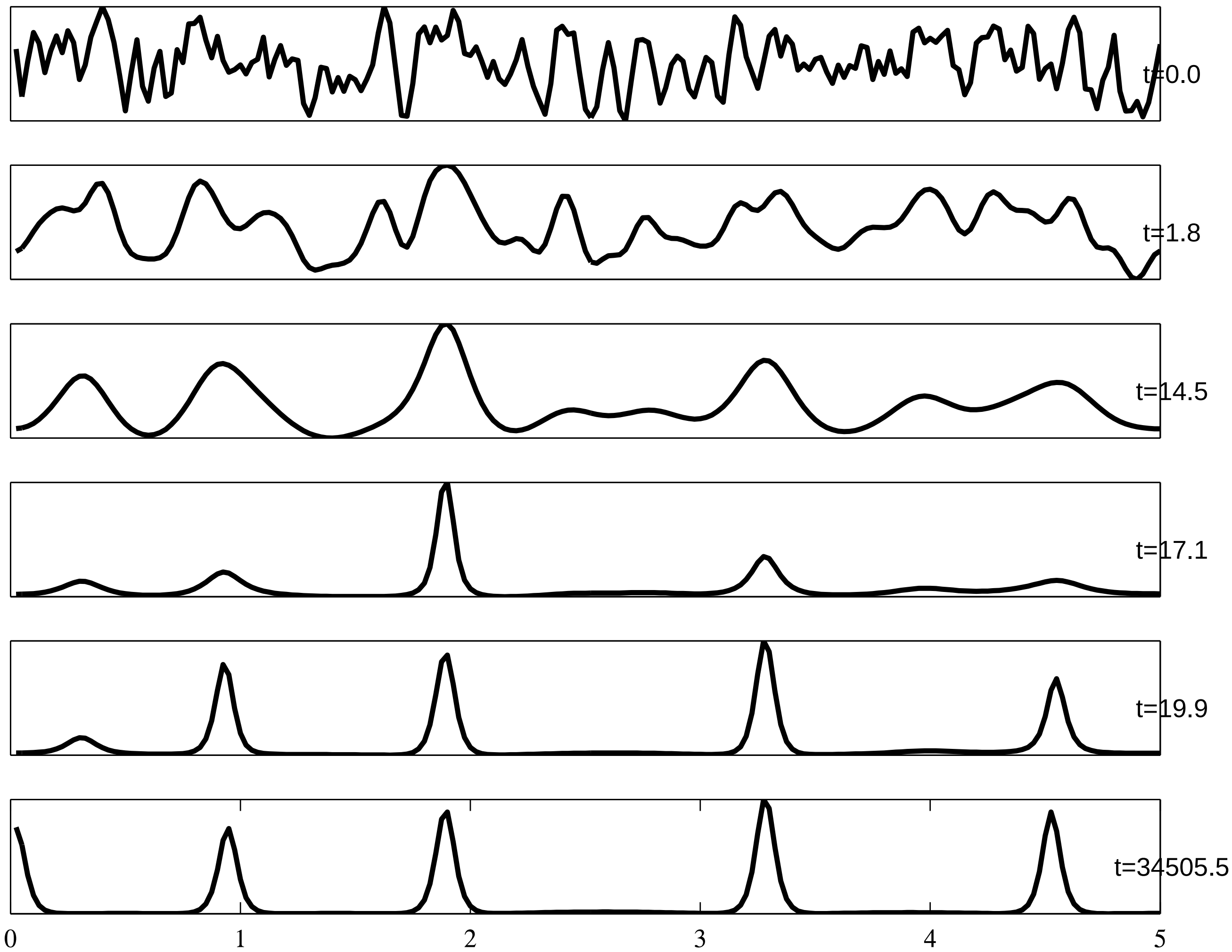
  - **Preferred Turing characteristic length:**

$$l_{\text{turing}} \sim 2\pi \left[\frac{D\eta}{\bar{\rho}(-2+3\bar{A}+6\bar{\rho})}\right]^{1/4}, \quad D\eta^{-1} \gg 1$$

- Note that both $O\left(l_c\right) = O(l_{turing}) = O((D\eta)^{1/4})$!

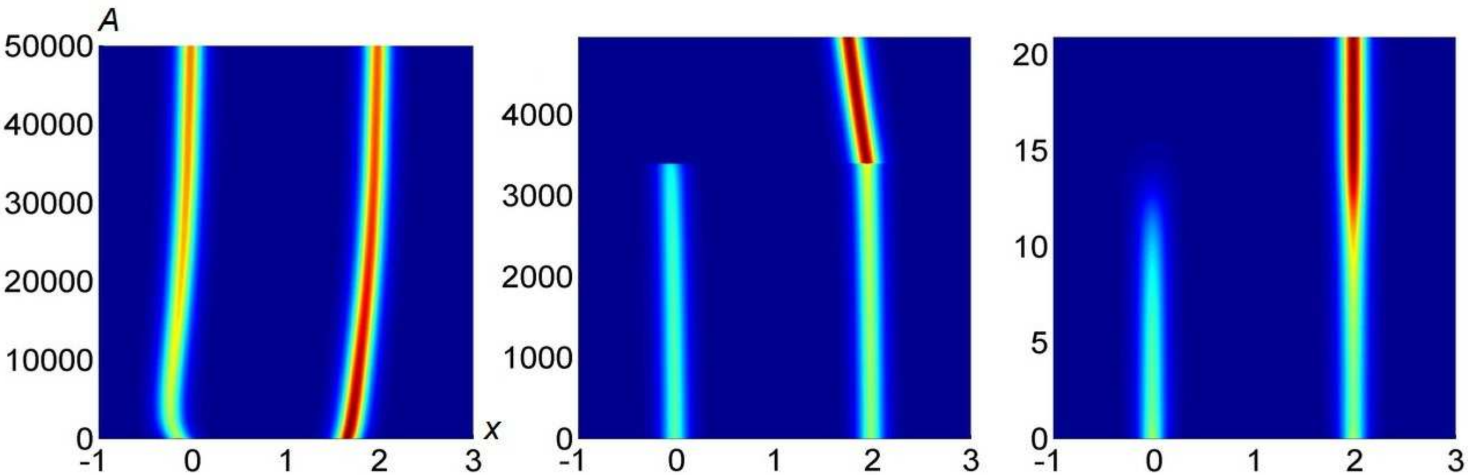
Example: $\alpha = 1,\ \ \gamma = 2,\ \ D = 1,\ \varepsilon = 0.03.$

Then $l_{turing} = 0.60; \quad l_c = 0.13 < l_{turing}$



Small and large eigenvalues

- Near-translational invariance leads to "small eigenvalues (perturbation from zero)" corresponding eigenfunction is $\phi \sim w'$.
- Large eigenvalues are responsible for "competition instability".
- Small eigenvalues become unstable before the large eigenvalues.
- Example: Take $l = 1, \gamma = 2, \alpha = 1,\ K = 2,\ \varepsilon = 0.07$. Then $D_{c,\text{small}} = 20.67,\ \ D_{c,\text{large}} = 41.33$.
  - if $D = 15 \implies$ two spikes are stable
  - if $D = 30 \implies$ two spikes have very slow developing instability
  - if $D = 50 \implies$ two spikes have very fast developing instability



Stability: large eigenvalues

- **Step 1:** Reduces to the nonlocal eigenvalue problem (NLEP):

$$\lambda\phi = \phi'' - \phi + 3w^2\phi - \chi\left(\int w^2\phi\right)w^3 \qquad \text{where } w'' - w + w^3 = 0. \tag{18}$$

with

$$\chi \sim \frac{3}{\int_{-\infty}^{\infty} w^3 dy}\left(1 + \varepsilon^2 D(1 - \cos\frac{\pi k}{K})\frac{\alpha^2\pi^2}{4l^4\beta^3}\right)^{-1}$$

- **Step 2:** ***Key identity***: $L_0 w^2 = 3w^2$, where $L_0\phi := \phi'' - \phi + 3w^2\phi$. Multiply (18) by w^2 and integrate to get

$$\lambda = 3 - \chi\int w^5 = 3 - \chi\frac{3}{2}\int w^3$$

Conclusion: (18) is stable iff $\chi > \frac{2}{\int w^3} \iff D > D_{c,\text{large}}$.

- This NLEP in 1D can be fully solved!!

Stability: small eigenvalues

- Compute asymmetric spikes

- They bifurcate from symmetric branch

- The bifurcation point is precisely when $D = D_{c,\text{small}}$.

- This is “cheating”... but it gets the correct threshold!!

Stability of K spikes

- Possible boundary conditions:

Config type	Boundary conditions for ϕ
Single interior spike on $[-l, l]$ even eigenvalue	$\phi'(0) = 0 = \phi'(l)$
Single interior spike on $[-l, l]$ odd eigenvalue	$\phi(0) = 0 = \phi'(l)$
Two half-spikes at $[0, l]$	$\phi'(0) = 0 = \phi(l)$
K spikes on $[-l, (2K-1)l]$, Periodic BC	$\phi(l) = z\phi(-l), \quad \phi'(l) = z\phi'(-l),$ $z = \exp(2\pi ik/K), \; k = 0 \ldots K-1$
K spikes on $[-l, (2K-1)l]$, Neumann BC	$\phi(l) = z\phi(-l), \quad \phi'(l) = z\phi'(-l),$ $z = \exp(\pi ik/K), \; k = 0 \ldots K-1$

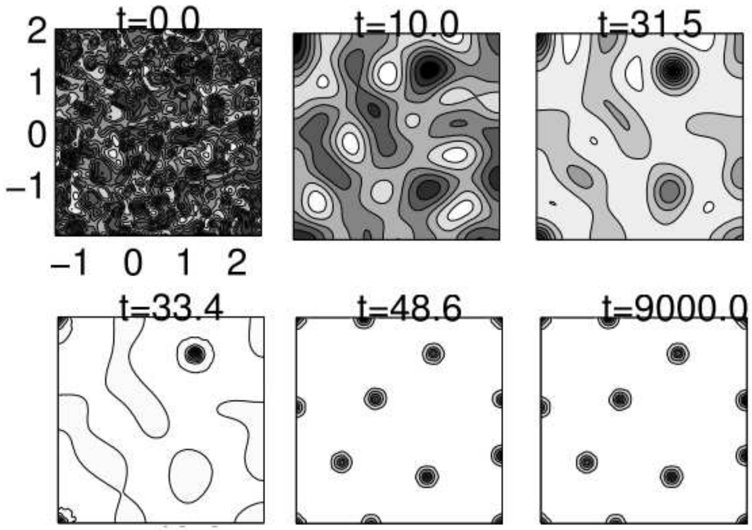
(same BC for ψ)

Two dimensions

Given domain of size S, let

$$K_c := 0.07037\eta^{-3/8}D^{-1/3}\left(\ln\frac{1}{\sqrt{\eta}}\right)^{1/3}\beta\alpha^{-2/3}S. \qquad (19)$$

Then K spikes are stable if $K < K_c$. Example: $\alpha = 1$, $\gamma = 2$, $\varepsilon = 0.08$, $D = 1$.



We get $S = 16$, $K_c \approx 10.19$. Starting with random initial conditions, the end state constits of $K = 7.5 < K_c$ hot-spots [counting boundary spots with weight $1/2$ and corner spots with weight $1/4$], in agreement with the theory.

Discussion

- Natural Separation of scales: $\eta^{-s} D \gg 1$
  - comes from the modelling assumptions
  - Required for hot-spot construction
  - The steady states are localized hotspots in the form of a sech, not sinusoidal bumps!
- Open question:
  - extend stability of hot-spots to Levi flights
  - More general moels of human motion?
- There is an optimal Levi flight exponent $1 < \mu < 3$ which "maximizes" the number of hot-spots. Do criminals "optimize" their strategy with respect to μ?
- References:

  - J. Breslau, T. Chaturapruek, D. Yazdi, S. McCalla and T. Kolokolnikov, *Incorporating Levi flights into a model of crime,* in preparation
  - T. Kolokolnikov, M. Ward and J. Wei, *The Stability of Steady-State Hot-Spot Patterns for a Reaction-Diffusion Model of Urban Crime,* to appear, DCDS-B.