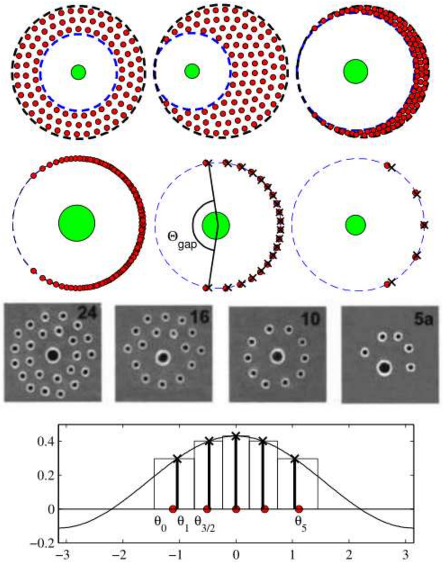
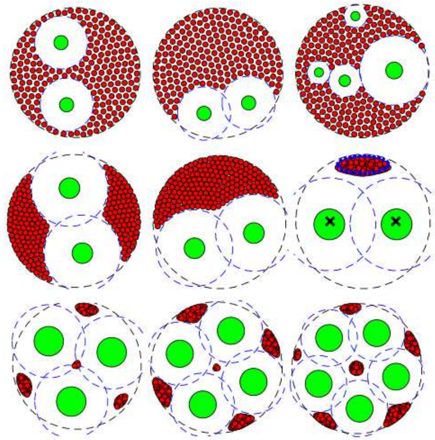
Vortex dynamics, animal skin patterns, and ice fishing





Theodore Kolokolnikov

Joint works with Michael Ward and Juncheng Wei, Yuxin Chen, Daniel Zhirov, Ricardo Carretero, Panoyatis Keverkedis

Vortex dynamics

- Equations first given by Helmholtz (1858): each vortex generates a rotational velocity field which advects all other vortices. ***Vortex model:***

$$\frac{dz_j}{dt} = i \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \ldots N.$$

- Classical problem; observed in many physical experiments: floating magnetized needles (Meyer, 1876); Malmberg-Penning trap (Durkin & Fajans, 2000), Bose-Einstein Condensates (Ketterle et.al. 2001); magnetized rotating disks (Whitesides et.al, 2001)

- Conservative, hamiltonian system

- General initial conditions lead to chaos: $movie - chaos$

- Certain special configurations are “stable” in hamiltonial sense: $movie - stable$

- Rigidly rotating steady states are called ***relative equilibria***:

$$z_j(t) = e^{\omega i t} \xi_j \quad \Longleftrightarrow \quad 0 = \sum_{k \neq j} \gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega \xi_j$$

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Dynamic, self-assembled aggregates of magnetized, millimeter-sized objects rotating at the liquid-air interface: Macroscopic, two-dimensional classical artificial atoms and molecules

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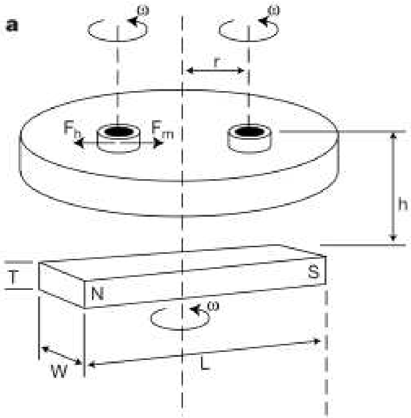


ure 1 Experimental set-up and magnetic force profiles. **a**, A scheme of the erimental set-up. A bar magnet rotates at angular velocity ω below a dish filled with iid (typically ethylene glycol/water or glycerine/water solutions). Magnetically doped ‹s are placed on the liquid–air interface, and are fully immersed in the liquid except for ir top surface. The disks spin at angular velocity ω around their axes. A magnetic force attracts the disks towards the centre of the dish, and a hydrodynamic force F_h pushes

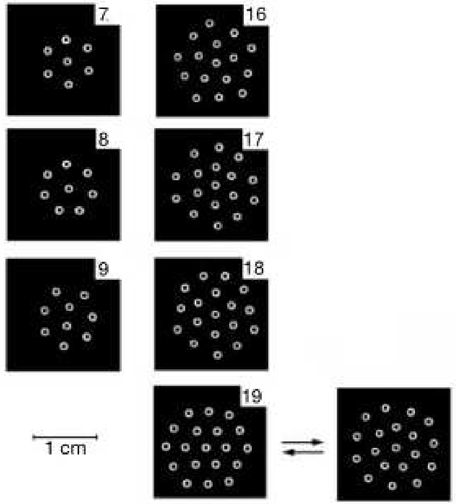


Figure 2 Dynamic patterns formed by various numbers (n) of disks rotating at the ethylene glycol/water–air interface. This interface is 27 mm above the plane of the external magnet. The disks are composed of a section of polyethylene tube (white) of outer diameter 1.27 mm, filled with poly(dimethylsiloxane), PDMS, doped with 25 wt% of magnetite (black centre). All disks spin around their centres at ω = 700 r.p.m., and the entire aggregate slowly ($\Omega < 2$ r.p.m.) precesses around its centre. For $n < 5$, the aggregates do not have a 'nucleus'—all disks are precessing on the rim of a circle. For $n > 5$, nucleated structures appear. For $n = 10$ and $n = 12$, the patterns are bistable in the sense that the two observed patterns interconvert irregularly with time. For $n = 19$, the hexagonal pattern (left) appears only above $\omega \approx 800$ r.p.m., but can be 'annealed' down

Observation of Vortex Lattices in Bose-Einstein Condensates

20 APRIL 2001 VOL 292 SCIENCE

J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle

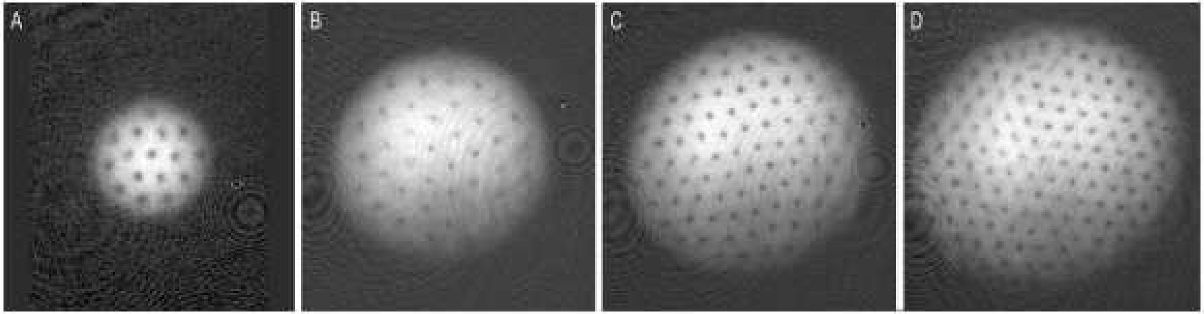
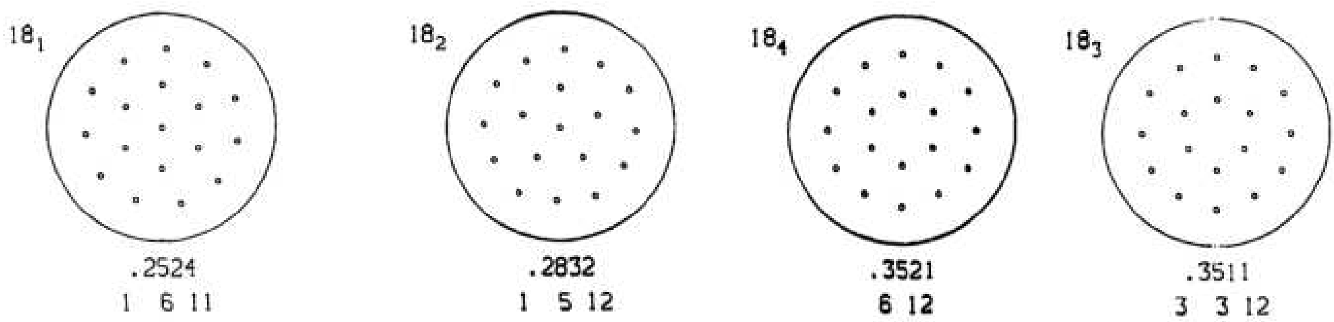


Fig. 1. Observation of vortex lattices. The examples shown contain approximately (**A**) 16, (**B**) 32, (**C**) 80, and (**D**) 130 vortices. The vortices have "crystallized" in a triangular pattern. The diameter of the cloud in (D) was 1 mm after ballistic expansion, which represents a magnification of 20. Slight asymmetries in the density distribution were due to absorption of the optical pumping light.

- Campbell and Ziff (1978) classified many stable configurations for ***small*** $(eg.\ N = 18)$ number of vortices of equal strength.



- Goal: describe the stable configuration in the continuum limit of a ***large*** number of vortices N (eg. $N = 100,\ 1000\ldots$). These have been observed in several recent expriments: Bose Einstein Condensates, magnetized disks

Key observation

$$\text{Vortex model:}\quad \frac{dz_j}{dt} = i\sum_{k\neq j}\gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1\ldots N. \tag{V}$$

$$\text{Relative equilibrium:}\quad z_j(t) = e^{\omega it}\xi_j \quad \Longleftrightarrow \quad 0 = \sum_{k\neq j}\gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega\xi_j$$

$$\text{Aggregation model:}\quad \frac{dx_j}{dt} = \sum_{k\neq j}\gamma_k \frac{x_j - x_k}{|x_j - x_k|^2} - \omega x_j. \tag{A}$$

- One-to-one correspondence between the steady statates $x_j(t) = \xi_j$ of (A) and the relative equilibrium $z_j(t) = e^{\omega it}\xi_j$ of (V).
- ***Spectral equivalence of (V) and (A):*** The equilibrium $x_j(t) = \xi_j$ is asymptotically stable for the aggregation model (A) if and only if the relative equilibrium $z_j(t) = e^{\omega it}\xi_j$ is stable (neutrally, in the Hamiltonian sense) for the vortex model (V)!
- Aggregation model fully describes relative equilibria and their linear stability in the vortex model.
- Aggregation model is easier to study than the vortex model.

Vortices of equal strength $\gamma_k = \gamma$

$$\frac{dz_j}{dt} = i\gamma \sum_{k \neq j} \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \dots N.$$

- In the limit $N \to \infty$, the steady state density of (A) is constant inside the ball of radius

$$R_0 = \sqrt{N\gamma/\omega}.$$

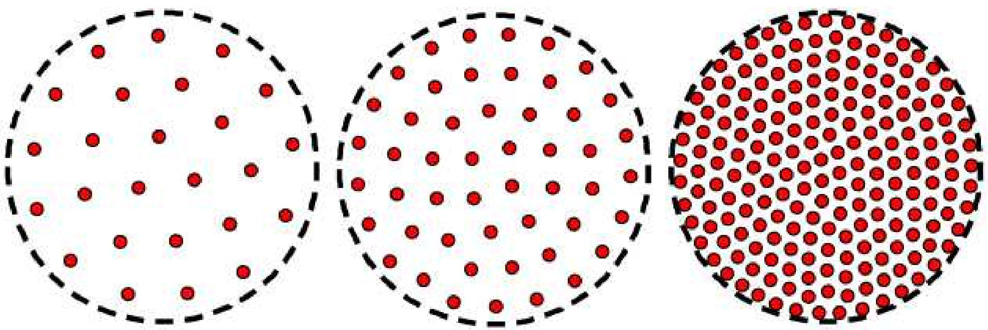
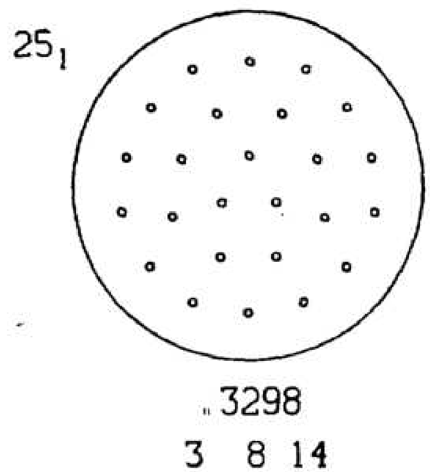


Fig. 1. Stable relative equilibria of $N = 25, 50$ and 200 vortices of equal strength. The dashed line shows the analytical prediction $R_0 = \sqrt{N\gamma/\omega}$ of the swarm radius in the $N \to \infty$ limit (see (6)).



Connection to the biological aggregation model

- [FHK,2011] Multi-particle interaction model:

$$\frac{dx_j}{dt} = \underbrace{\frac{1}{N}\sum_{k\neq j}\frac{x_j - x_k}{|x_j - x_k|^2}}_{Newtonian\ repulsion} \quad - \quad \underbrace{x_j}_{Linear\ attraction} \quad j = 1\dots N. \tag{1}$$

(2)

- This is just the first two terms of the ice-fishing problem (no reflection in the boundary)
- This model results in a **constant density swarm.**

- ***Newtonian*** repulsion, ***linear*** attraction.
- In the limit $N \to \infty$, the density is constant inside a ball of radius 1; zero outside.

Continuum limit

- We define the ***density*** ρ as

$$\int_D \rho(x)dx \approx \frac{\#\text{particles inside domain } D}{N}$$

- The flow is then characterized by density ρ and velocity field v:

$$\rho_t + \nabla \cdot (\rho v) = 0; \qquad v(x) = \int_{\mathbb{R}^n} \left(\frac{x-y}{|x-y|^2} - x - y \right) \rho(y)dy. \tag{3}$$

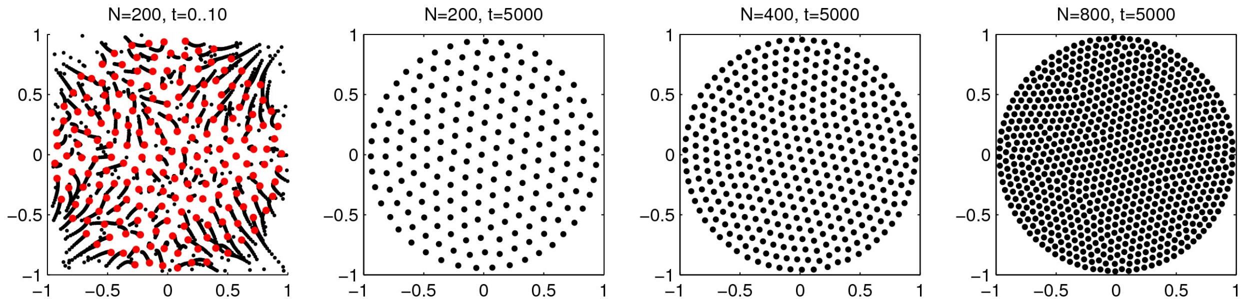
- We have

$$v(x) = \int \nabla_x \left(\log|x-y| - \frac{1}{2}|x-y|^2 \right) \rho(y)dy$$

$$\begin{aligned} \nabla \cdot v &= \int (2\pi\delta(x-y) - 2)\,\rho(y)dy \\ &= 2\pi\rho(x) - 2M \end{aligned}$$

- Inside, the swarm, $\nabla \cdot v = 0 \implies \rho = M/\pi$ is constant!

- Radius is determined by conservation of mass: $M = \rho \pi R^2 \implies R = 1.$



$N+1$ problem

- N vortices of equal strength and a single vortex of a much higher strength:

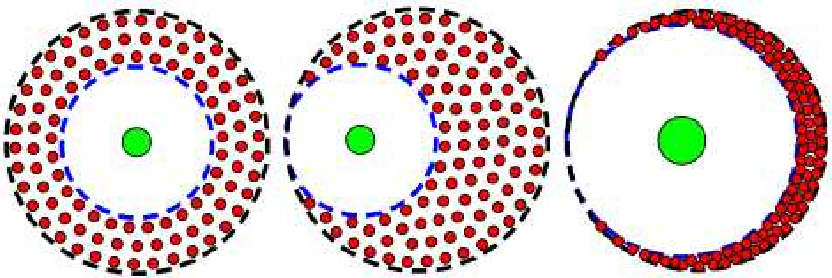
$$\frac{dx_j}{dt} = \frac{a}{N} \sum_{\substack{k=1\ldots N \\ k \neq j}} \frac{x_j - x_k}{|x_j - x_k|^2} + b\frac{x_j - \eta}{|x_j - \eta|^2} - x_j, \quad j = 1 \ldots N, \tag{4}$$

$$\frac{d\eta}{dt} = \frac{a}{N} \sum_{k=1\ldots N} \frac{\eta - x_k}{|\eta - x_k|^2} - \eta \tag{5}$$

- Mean-field limit $N \to \infty$:

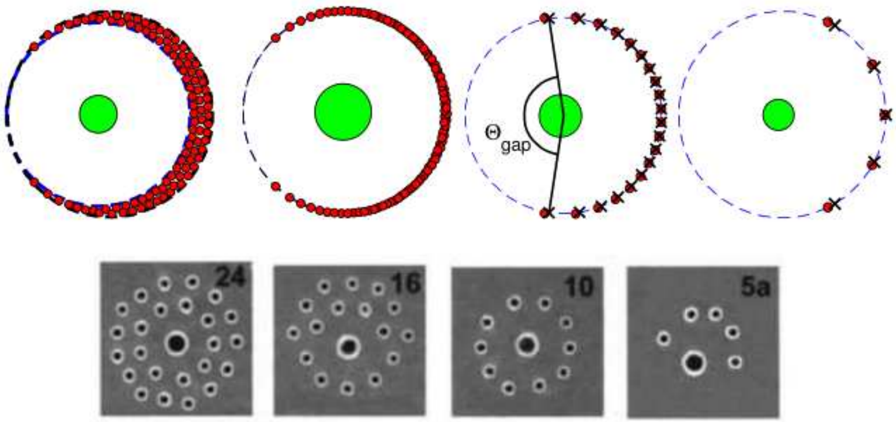
$$\begin{cases} \rho_t + \nabla \cdot (\rho \nabla v) = 0; \\ v(x) = a \int_{\mathbb{R}^2} \rho(y) \frac{x-y}{|x-y|^2} dy + b\frac{x-\eta}{|x-\eta|^2} - x \\ \frac{d\eta}{dt} = a \int_{\mathbb{R}^2} \rho(y) \frac{\eta - y}{|\eta - y|^2} dy - \eta \end{cases}. \tag{6}$$

- **Main result:**. Define $R_1 = \sqrt{b}$, $R_0 = \sqrt{a+b}$ and suppose that η is any point such that $B_{R_1}(\eta) \subset B_{R_0}(0)$. Then the equilibrium solution for (6) is constant inside $B_{R_0}(0) \backslash B_{R_1}(\eta)$ and is zero outside.



- Unlike the $N+0$ problem, the relative equilibrium for the $N+1$ problem is non-unique: any choice of η yields a steady state as long as $|\eta| < R_0 - R_1$.

Degenerate case: big central vortex



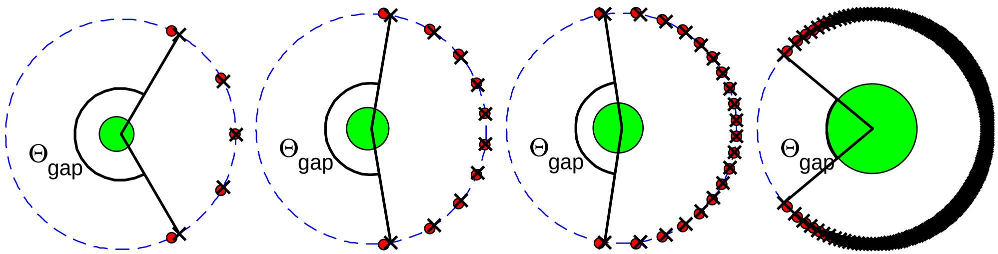
- Small vortices are constrained to a ring of radius R_0. with big vortex at the center.
- ***Non-uniform*** distribution of small particles!
- Question: Determine the size of the gap Θ_{gap}.

- **Main Result:**

$$\Theta_{gap} \sim CN^{-1/3}.$$

where the constant $C = 8.244$ satisfies

$$\left(8 - 6u + 2u^3\right) \ln\left(u - 1\right) = 3u\left(u^2 - 4\right); \quad C = 2\left(\frac{6\pi(2-u)}{u\left(u^2-1\right)}\right)^{1/3}$$



Sketch of proof

- [Barry+Wayne, 2012]: Set $x_j(t) \sim R_0 e^{i\theta_j(t)}$ then at leading order we get

$$\frac{d\theta_j}{dt} = \frac{1}{N}\sum_{k\neq j}\left(\frac{\sin(\theta_j-\theta_k)}{2-2\cos(\theta_j-\theta_k)} - \sin(\theta_j-\theta_k)\right). \tag{7}$$

- In the mean-field limit $N \to \infty$, the density distribution $\rho(\theta)$ for the angles θ_j satisfies

$$\begin{cases} \rho_t + (\rho v_\theta)_\theta = 0, \\ v(\theta) = PV\displaystyle\int_{-\pi}^{\pi}\rho(\phi)\left(\frac{\sin(\theta-\phi)}{2-2\cos(\theta-\phi)} - \sin(\theta-\phi)\right)d\phi, \end{cases} \tag{8}$$

  where PV denotes the principal value integral, and $\int_{-\pi}^{\pi}\rho = 1$.

- [Barry, PhD Thesis]: Up to rotations, the steady state density $\rho(\theta)$ for which $v = 0$ must be of the form

$$\rho(\theta) = \frac{1}{2\pi}(1 + \alpha\cos\theta). \tag{9}$$

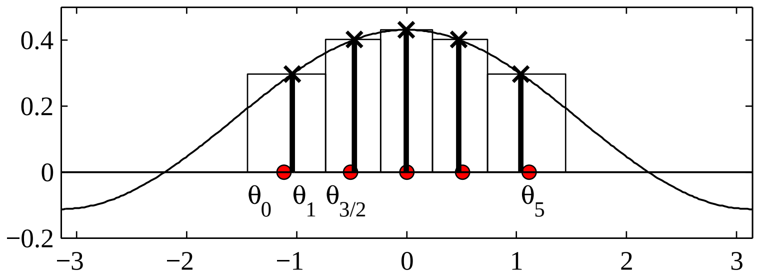
  This follows from (8) and (formal) expansion

$$\frac{\sin t}{2-2\cos t} - \sin t = \sin(2t) + \sin(3t) + \sin(4t) + \ldots$$

- α is free parameter in the continuum limit.

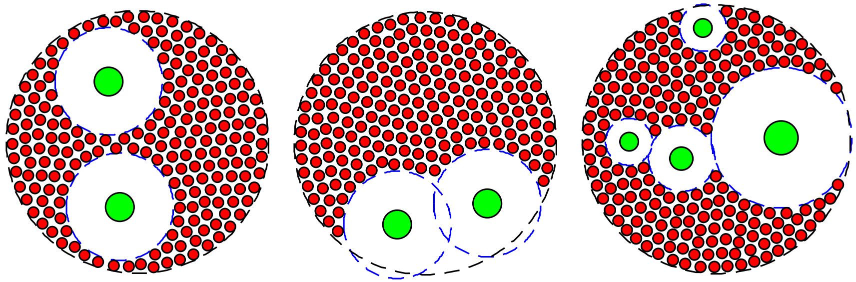
- For discrete N, particle positions satisfy

$$\int_{\theta_{j-1}}^{\theta_j} \frac{1}{2\pi} (1 + \alpha \cos \theta) \, d\theta = \frac{1}{N}$$



To estimate Φ_{gap}, choose θ_1 so that $v(\theta_1) \sim 0$. See our paper for hairy details.

$N+K$ problem



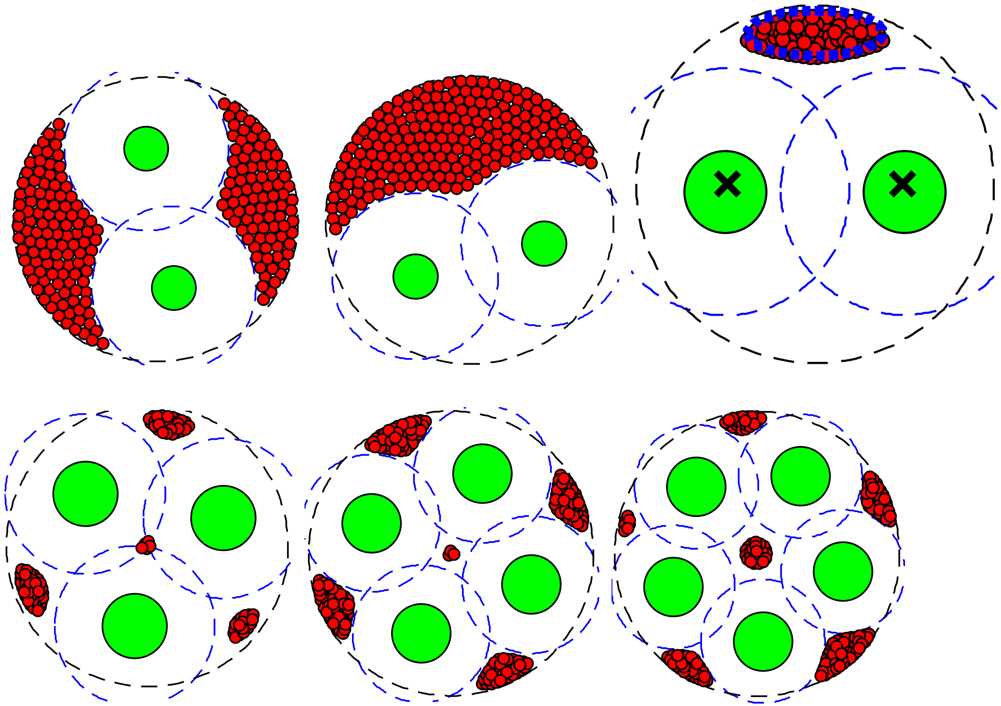
$$v(x) = a\int_{\mathbb{R}^2} \rho(y)\,\frac{x-y}{|x-y|^2}dy + \sum_{k=1\ldots K} b_k \frac{x-\eta_k}{|x-\eta_k|^2} - x,$$

$$\frac{d\eta_j}{dt} = a\int_{\mathbb{R}^2} \rho(y)\,\frac{\eta_k-y}{|\eta_k-y|^2}dy + \sum_{\substack{k=1\ldots K\\ k\neq j}} b_k \frac{\eta_j-\eta_k}{|\eta_j-\eta_k|^2} - \eta_j,$$

$$j = 1\ldots K.$$

Main result: Let $R_k = \sqrt{b_k}, \;\; k = 1\ldots K$ and $R_0 = \sqrt{a + b_1 + \ldots + b_K}$. Suppose $\eta_1 \ldots \eta_K$ are such $B_{R_1}(\eta_1)\ldots B_{R_K}(\eta_K)$ are all disjoint and are contained inside $B_{R_0}(0)$. The equilibrium density is constant inside $B_{R_0}(0)\backslash \bigcup_{k=1}^{K} B_{R_k}(\eta_k)$ and is zero outside.

$N + K$ problem, with very large K vortices



- The ***blue ellipse*** is described by the reduced system

$$\frac{d\xi_j}{dt} = \frac{1}{N} \sum_{\substack{k=1\ldots N \\ k\neq j}} \frac{1}{\overline{\overline{\xi_j - \xi_k}}} + \frac{1}{2}\bar{\xi}_k - \xi_k \tag{10}$$

- From [K, Huang, Fetecau, 20011], its axis ratio is 3.

Crystallization

$$\text{Vortex model: } \frac{dz_j}{dt} = i \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2}, \quad j = 1 \ldots N. \tag{V}$$

$$\text{Reltive equiliria: } z_j(t) = e^{\omega i t} \xi_j \quad \Longleftrightarrow \quad 0 = \sum_{k \neq j} \gamma_k \frac{\xi_j - \xi_k}{|\xi_j - \xi_k|^2} - \omega \xi_j$$

$$\text{Vortex with dissipation: } \frac{dz_j}{dt} = i \sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2} + \mu \left(\sum_{k \neq j} \gamma_k \frac{z_j - z_k}{|z_j - z_k|^2} - \omega z_j \right) \tag{D}$$

- In many physical experiments of BEC there is damping or dissipation involved.
- ***Spectral equivalence:*** Relative equilibria ***and their stability*** are the same for (V) and (D)
- Both the vortex model and the "aggregation model" model are limiting cases of (D).
- Taking $\mu > 0$ ***stabilizes vortex dynamics!*** *chaos damped stable*
- This allows us to find stable relative equilibria numerically.

Vortex dynamics in BEC with trap

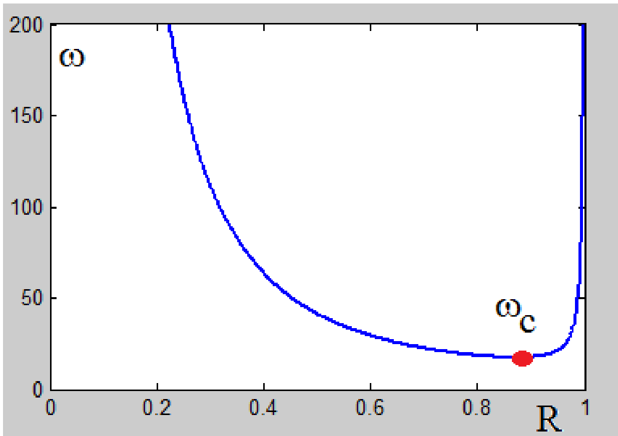
- For BEC, dynamics have extra term corresponding to prcession around the trap:

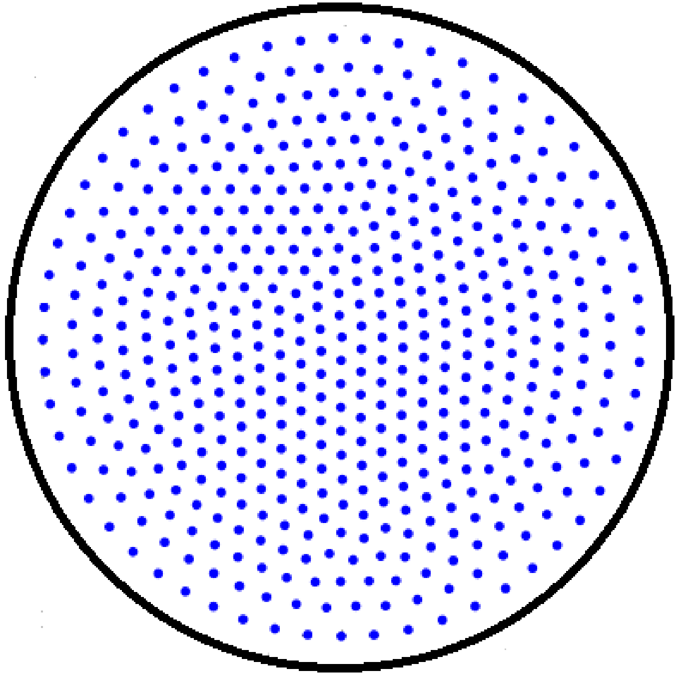
$$\dot{z}_j = i\underbrace{\frac{a}{1-r^2}}_{\text{trap-interaction}} z_j + \underbrace{ic\sum_{k\neq j}\frac{z_j - z_k}{|z_j - z_k|^2}}_{\text{self-interaction}}, \quad j = 1\ldots N. \tag{11}$$

- Large N limit: ***non-uniform*** vortex lattice:

$$\rho \sim \omega - \frac{a}{(1-r^2)^2} \text{ if } r < R, \quad \rho = 0 \text{ otherwise,}$$

with $\omega = \frac{a}{1-R^2} + \frac{cN}{R^2}$





$$\omega_c = \left(\sqrt{a} + \sqrt{cN}\right)^2; \quad R_c^2 = \frac{\sqrt{cN}}{\sqrt{a} + \sqrt{cN}}.$$

- No solutions ofr $\omega < \omega_c$
- Two solutions $R = R_\pm$ if $\omega > \omega_c$, smaller is stable, larger unstable.

N-body problem

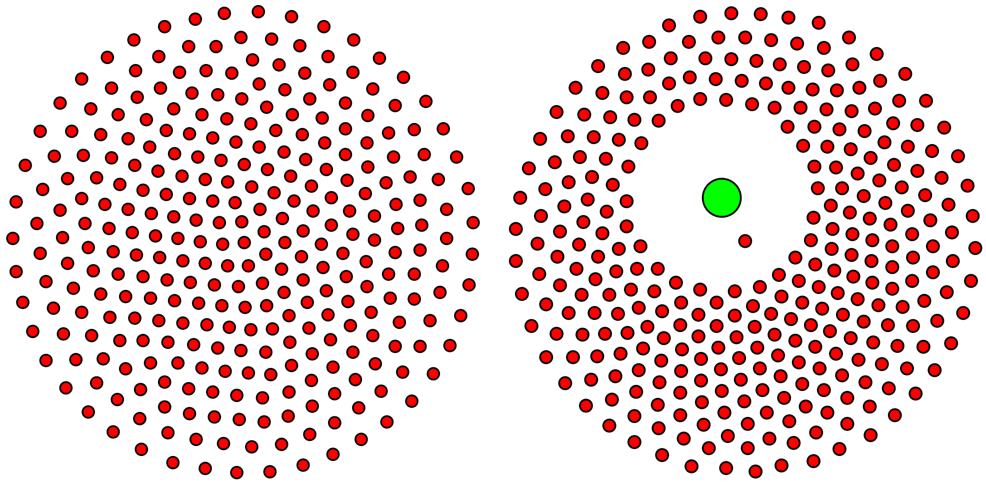
$$\ddot{z}_j = \sum_{k \neq j} c_k c_j \frac{z_k - z_j}{|z_k - z_j|^{3}} \tag{12}$$

- Relative equilibria $z_j = e^{i\omega t} x_j$ satisfy:

$$0 = \sum_{k \neq j} c_k c_j \frac{x_k - x_j}{|x_k - x_j|^3} + \omega^2 x_j \tag{13}$$

- Gradient flow (to find steady states):

$$-\dot{x}_j = \sum_{k \neq j} c_k c_j \frac{x_k - x_j}{|x_k - x_j|^3} + \omega^2 x_j \tag{14}$$



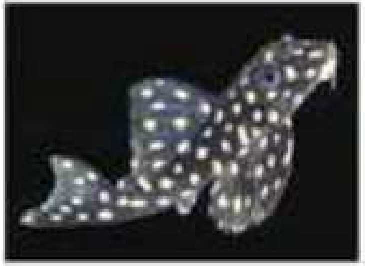
relative equilibrium for 300-body problem (unstable)

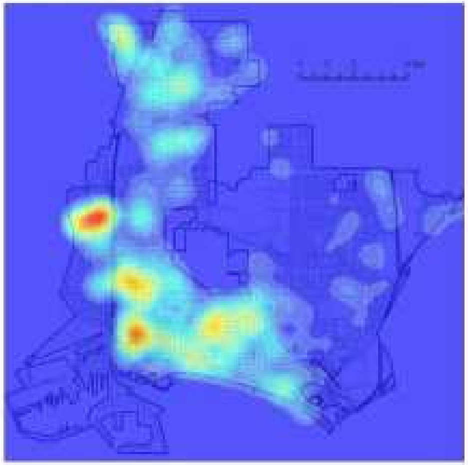
- For N equal-mass bodies, the relative equilibrium is known to be unstable when $N \geq 3$.

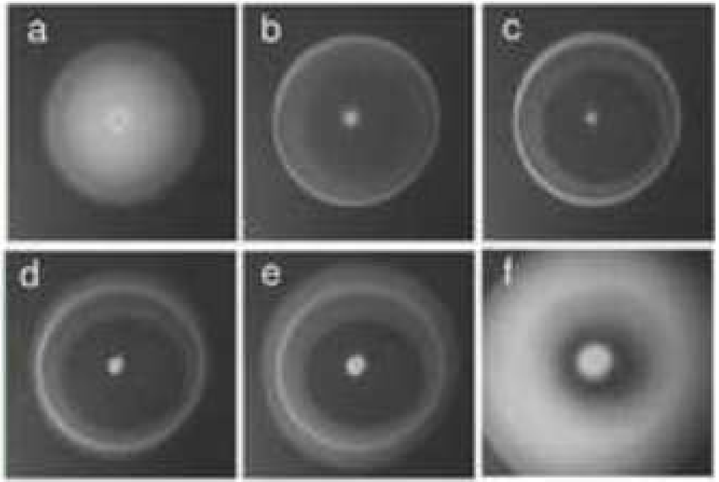
- Unlike the vortex model, there is ***no spectral equivalence between (12) and (14)***

Spot solutions in Reaction-diffusion systems

seashells * fish * crime hotspots in LA * stressed bacterial colony





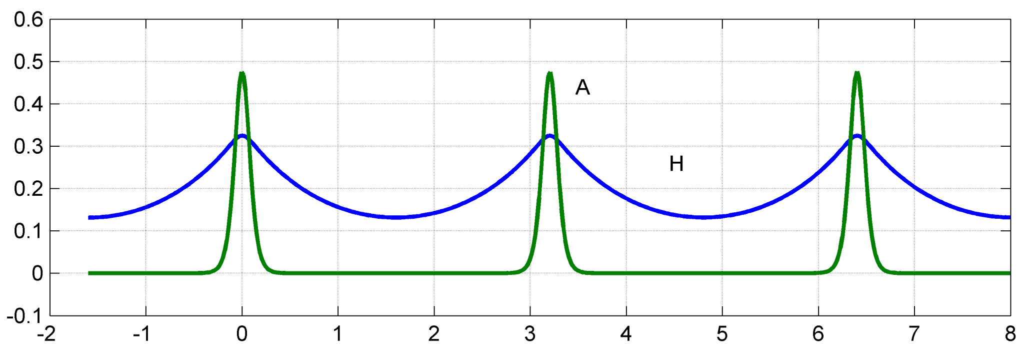


Classical Gierer-Meinhardt model

$$A_t = \varepsilon^2 \Delta A - A + \frac{A^2}{H}; \qquad \tau H_t = D \Delta H - H + A^2$$

- Introduced in 1970's to model cell differentation in hydra
- Mostly of mathematical interest: one of the simplest RD systems
- Has been intensively studied since 1990's [by mathematicians!]
- Key assumption: ***separation of scales***

$$\varepsilon \ll 1 \text{ and } \varepsilon^2 \ll D.$$



- Roughly speaking, H is constant on the scale of A so the steady state looks ”roughly” like $A(x) \sim Cw\left(\frac{x - x_0}{\varepsilon}\right)$ where

$$\Delta w - w + w^2 = 0.$$

- Questions: What about stability? What about location of the spike x_0?

“Classical” Results in 1D:

- Wei 97, 99, Iron+Wei+Ward 2000: Stability of K spikes in the GM model in one dimension

- Two types of possible instabilitities: structural instabilities or translational instabilities

- Structural instabilities (large eigenvalues) lead to spike collapse in $O(1)$ time

- Translational instabilities can lead to ”slow death”: spikes drift over large time scales

- **Main result 1**: There exists a sequence of thresholds D_K such that K spikes are stable iff $D < D_K$.

- **Main result 2:** Slow dynamics of K spikes is described by an ODE with $2K$ variables (spike heights and centers) subject to K algebraic constraints between these variables.

Large eigenvalues

- Careful derivation leads to a ***nonlocal eigenvalue problem*** (NLEP) of the form

$$\lambda\phi = \Delta\phi + (-1+2w)\,\phi - \chi w^2 \frac{\int w\phi}{\int w^2}; \quad \chi := \frac{4\sinh^2\left(\frac{1}{\sqrt{D}}\right)}{2\sinh^2\left(\frac{1}{\sqrt{D}}\right) + 1 - \cos\left[\pi(1-1/K)\right]}$$

- **Key theorem (Wei, 99):** $\mathrm{Re}(\lambda) < 0$ iff $\chi < 1$

- **Corrollary**: On a domain $[-1,1]$, large eigenvalues are stable iff $D < D_{K,\text{large}}$ where

$$D_{K,\text{large}} = \frac{1}{\operatorname{arcsinh}^2(\sin 2\pi/K)}$$

- When unstable, this can lead to ***competition instability.***

- Movies: stable; unstable

Small eigenvalues

- Causes a ***very slow drift***

- Iron-Ward-Wei 2000: The slow dynamics of the system can be reduced to a coupled algbraic-differential system of ODEs

- Movie: slow drift

Two dimensions

- Structural stability is similar
- Dynamics [Ward et.al, 2000, K-Ward, 2004, K-Ward 2005]:

$$\frac{dx_0}{dt} \sim -\frac{4\pi\varepsilon^2}{\ln \varepsilon^{-1} + 2\pi R_0} \nabla R_0$$

where

$$R_0 = \lim_{x \to x_0} \left[G(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right];$$
$$\nabla R_0 = \lim_{x \to x_0} \nabla_x \left[G(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right];$$

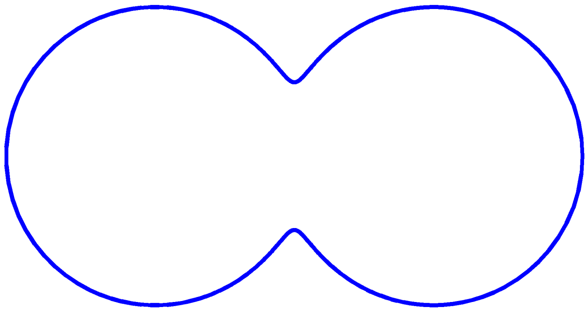
$$\Delta G - \frac{1}{D} G = -\delta\left(x - x_0\right) \text{ on } \Omega; \quad \partial_n G = 0 \text{ on } \partial\Omega$$

- Equilibrium location x_0 satisfies $\nabla R_0 = 0$, occurs at the extremum of the regular part of the Neumann's Green's function

Dumbbell-shaped domain

- QUESTION: Suppose that a domain has a dumb-bell shape. Where will the spike drift??

- What are the possible equilibrium locations for a single spike?



Small D limit

- If D is very small, $R_0(x_0) \sim C(x_0) \exp\left(-\frac{1}{\sqrt{D}}|x_0 - x_m|\right)$ where x_m is the point on the boundary closest to x_0

- This means that R_0 is ***minimized at the point furthest away from the boundary when*** $D \ll 1$
  - In the limit $\varepsilon^2 \ll D \ll 1$, the spike drifts towards the point furthest away from the boundary.
  - For a dumbell-shaped domain above, the three possible equilibria are at the ”centers” of the dumbbells (stable) and at the center of the neck (unstable saddle point)
  - For multiple spikes, their locations solve ”ball-packing problem”.

- Movie: $D = 0.03, \varepsilon = 0.04$

Large D limit

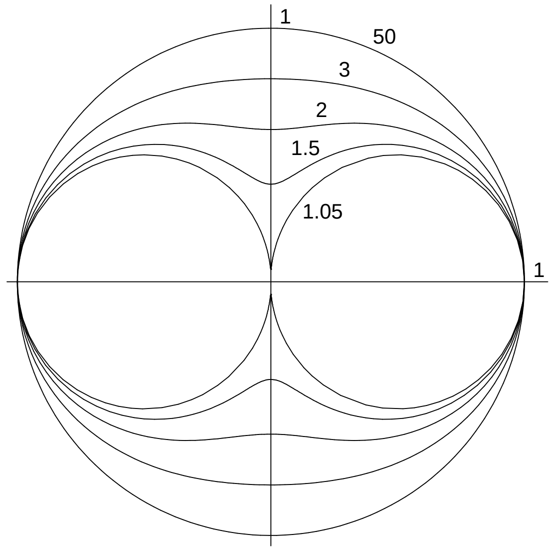
- We get the ***modified Green's function:***

$$\Delta G_m - \frac{1}{|\Omega|} = -\delta(x - x_0) \text{ inside } \Omega, \quad \partial_n G = 0 \text{ on } \partial\Omega;$$

$$R_{m0} = \lim_{x \to x_0} \left[G_m(x, x_0) + \frac{1}{2\pi} \ln(|x - x_0|) \right].$$

- [K, Ward, 2003]: For a domain which is an analytic mapping of a unit disk, $\Omega = f(B)$, we derive an ***exact formula*** for ∇R_{m0} in terms of the residues of $f(z)$ *outside* the unit disk.

- Take $f(z) = \dfrac{(1 - a^2)z}{z^2 + a^2}; \qquad x_0 = f(z_0):$



Then

$$\nabla R_{m0}(x_0) = \frac{\nabla s(z_0)}{f'(z_0)}$$

where

$$\nabla s(z_0) = \frac{1}{2\pi} \left(\begin{array}{c} \frac{z_0}{1-|z_0|^2} - \frac{\left(\bar{z}_0^2+3a^2\right)\bar{z}_0}{\bar{z}_0^4-a^4} + \frac{a^2\bar{z}_0}{\bar{z}_0^2a^2-1} + \frac{\bar{z}_0}{\bar{z}_0^2-a^2} \\ -\frac{(a^4-1)^2(|z_0|^2-1)(z_0+a^2\bar{z}_0)(\bar{z}_0^2+a^2)}{(a^4+1)(\bar{z}_0^2a^2-1)(z_0^2-a^2)(\bar{z}_0^2-a^2)^2} \end{array} \right)$$

- Corrollary: for above Ω, ∇R_{m0} has a unique root at the origin!
  - In the limit $D \gg 1$, all spikes will drift towards the neck.
- Complex bifurcation diagram as D is increased.
- Movie: $\varepsilon = 0.05$, $D = 0.1$; $D = 1$.

”Huge” D

- In the limit $D \to \infty$, (Shadow limit), an interior spike is unstable and moves towards the boundary [Iron Ward 2000; Ni, Polácik, Yanagida, 2001].

- For ***exponentially large but finite*** $D = O(\exp(-C/\varepsilon))$, boundary effects will compete with the Green’s function.

- [K, Ward, 2004]: Define

$$\sigma := \frac{\varepsilon}{2} \ln \left(\frac{C_0}{|\Omega|} D \varepsilon^{-1/2} \right); \quad C_0 \approx 334.80;$$

  Then the spike will move towards the boundary whenever its distance from the closest point of the boundary is at most σ; otherwise it will move away from the boundary.

- Movies: $\varepsilon = 0.05$, $D = 10$; $D = 100$

Spike dynamics inside a disk

In the limit $\varepsilon \ll 1, D \gg 1$, inside the disk we get

$$C\frac{dx_j}{dt} \sim \underbrace{2\sum_{k\neq j}\frac{x_j - x_k}{|x_j - x_k|^2} - \sum_k x_j}_{inter-particle\ force} + \underbrace{\sum_k \frac{x_j - x_k/|x_k|^2}{\left|x_j - x_k/|x_k|^2\right|^2} - \sum_k \frac{-x_j|x_k|^2 + x_k|x_j|^2}{\left|x_j|x_k|^2 - x_k\right|^2}}_{reflection\ in\ the\ boundary\ of\ unit\ disk}.$$

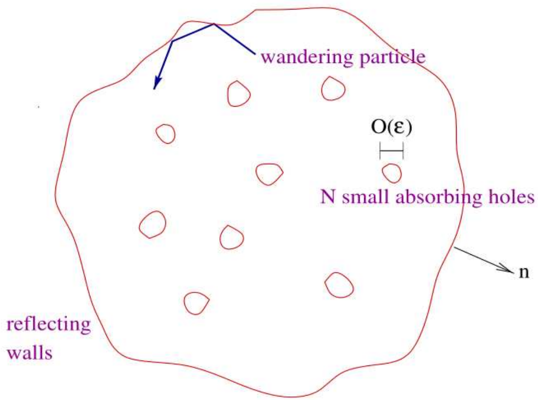
- The first two terms are identical to vortex stability model!
- The last two terms represent “reflection in the wall”
- Just like for vortex model, ***the steady state consists of uniformly-distributed particles inside the domain!***
- Movies: disk; dumbbell.

Mean first passage time (ice fishing)

- Question: Suppose you want to catch a fish in a lake covered by ice. Where do you drill a hole to maximize your chances?

- Related questions: cell signalling; oxygen transport in muscle tissues; cooling rods in a nuclear reactor...

- Consider N non-overlapping small "holes" each of small radius ε. A particle is performing a random walk inside the domain Ω. If it hits a hole, it gets destroyed; if it hits a boundary, it gets reflected. Question: what is the expected lifetime of the wondering particle? How do we place the holes to minimize this lifetime [i.e. catch the fish, cool the nuclear reactor...]?



- The expected lifetime is proportional to $1/\lambda$ where λ is the smallest eigenvalue of the problem:

$$\Delta u + \lambda u = 0 \text{ inside } \Omega \backslash \Omega_p; \quad u = 0 \text{ on } \partial\Omega_p;\ \partial_n u = 0 \text{ on } \partial\Omega$$

where $\Omega_p = \bigcup_{i=1}^{N} \Omega_\varepsilon$.

- [K-Ward-Titcombe, 2005]: The smallest eigenvalue is given by

$$\lambda \sim \frac{2\pi N}{\ln \frac{1}{\varepsilon}} \left(1 - \frac{2\pi}{\ln \frac{1}{\varepsilon}} p(x_1, \ldots x_N) + O\left(\frac{1}{\left(\ln \frac{1}{\varepsilon}\right)^2} \right) \right)$$

where

$$p(x_1, \ldots x_N) := \sum \sum G_{ij};$$

$$G_{ij} = \begin{cases} G_m(x_i, x_j) & \text{if } i \neq j \\ R_m(x_i, x_i) & \text{if } i = j \end{cases}$$

$$\Delta G_m(x, x') - \frac{1}{|\Omega|} = -\delta(x - x') \text{ inside } \Omega, \quad \partial_n G = 0 \text{ on } \partial\Omega; \quad R_m \equiv reg.part$$

- For a unit disk:

$$2\pi G_m(x, x') = -\ln |x - x'| - \ln \left| x\,|x'| - \frac{x'}{|x'|} \right| + \frac{1}{2}\left(|x|^2 + |x'|^2 \right)$$

$$2\pi R_m(x, x') = -\ln \left| x\,|x'| - \frac{x'}{|x'|} \right| + \frac{1}{2}\left(|x|^2 + |x'|^2 \right)$$

- The optimum trap placement is at the minimum of $p(x_1, \ldots x_N)$

Disk domain, N holes

We need to minimize

$$p(x_1 \ldots x_N) = -\sum_{j\neq k} \ln |x_j - x_k| - \sum_{j,k} \left(\ln \left| x_j - \frac{x_k}{|x_k|^2} \right| + \ln |x_k| \right) + \frac{1}{2} \sum_{j,k} \left(|x_j|^2 + |x_k|^2 \right)$$

Gradient flow is uniform swarm model plus two extra terms

$$\frac{dx_j}{dt} = 2\sum_{k\neq j} \frac{x_j - x_k}{|x_j - x_k|^2} - \sum_k x_j + \sum_k \frac{x_j - x_k/|x_k|^2}{\left|x_j - x_k/|x_k|^2\right|^2} - \sum_k \frac{-x_j |x_k|^2 + x_k |x_j|^2}{\left|x_j |x_k|^2 - x_k\right|^2}.$$

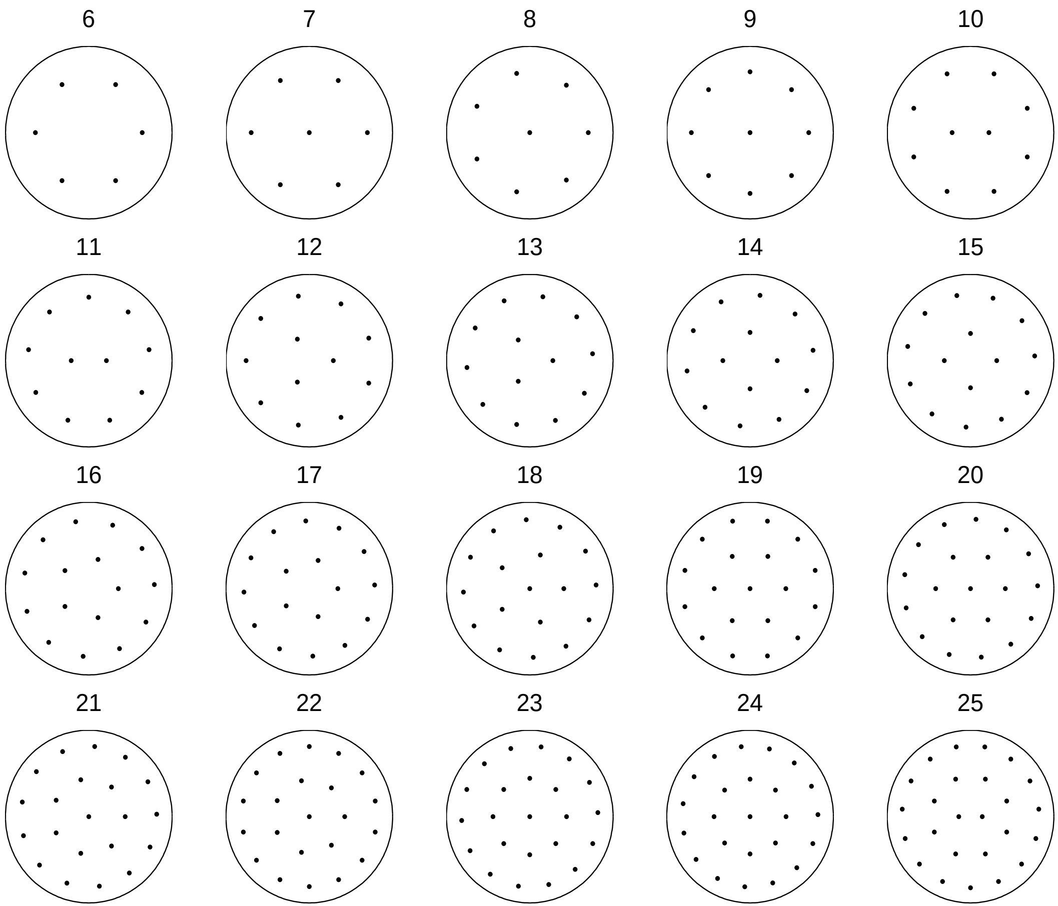
Particles on a ring: $x_k = re^{ik2\pi/N}$. The min occurs when

$$\frac{r^{2N}}{1 - r^{2N}} = \frac{N-1}{2N} - r^2$$

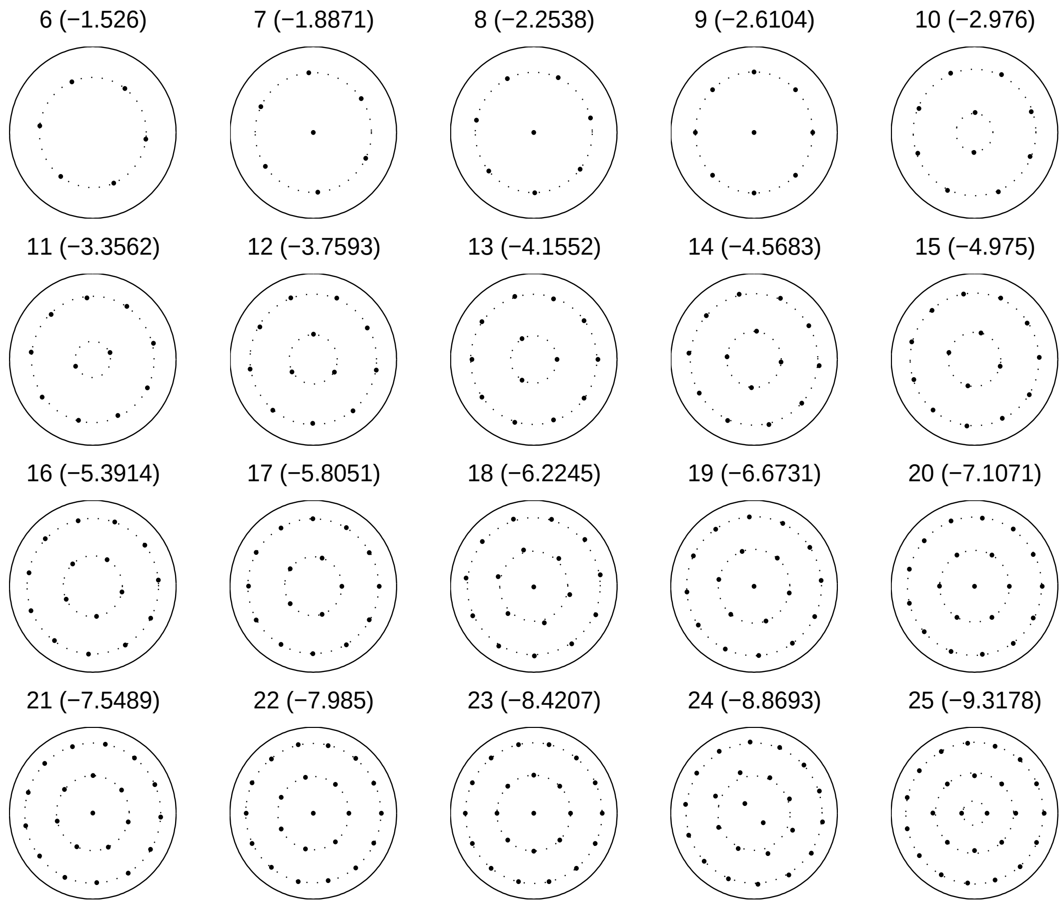
Note that $r \to 1/\sqrt{2}$ as $N \to \infty$; the optimal ring divides the unit disk into two equal areas.

Particles on 2,3,... m rings: Similar results are derived with complicated but numerically useful formulas.

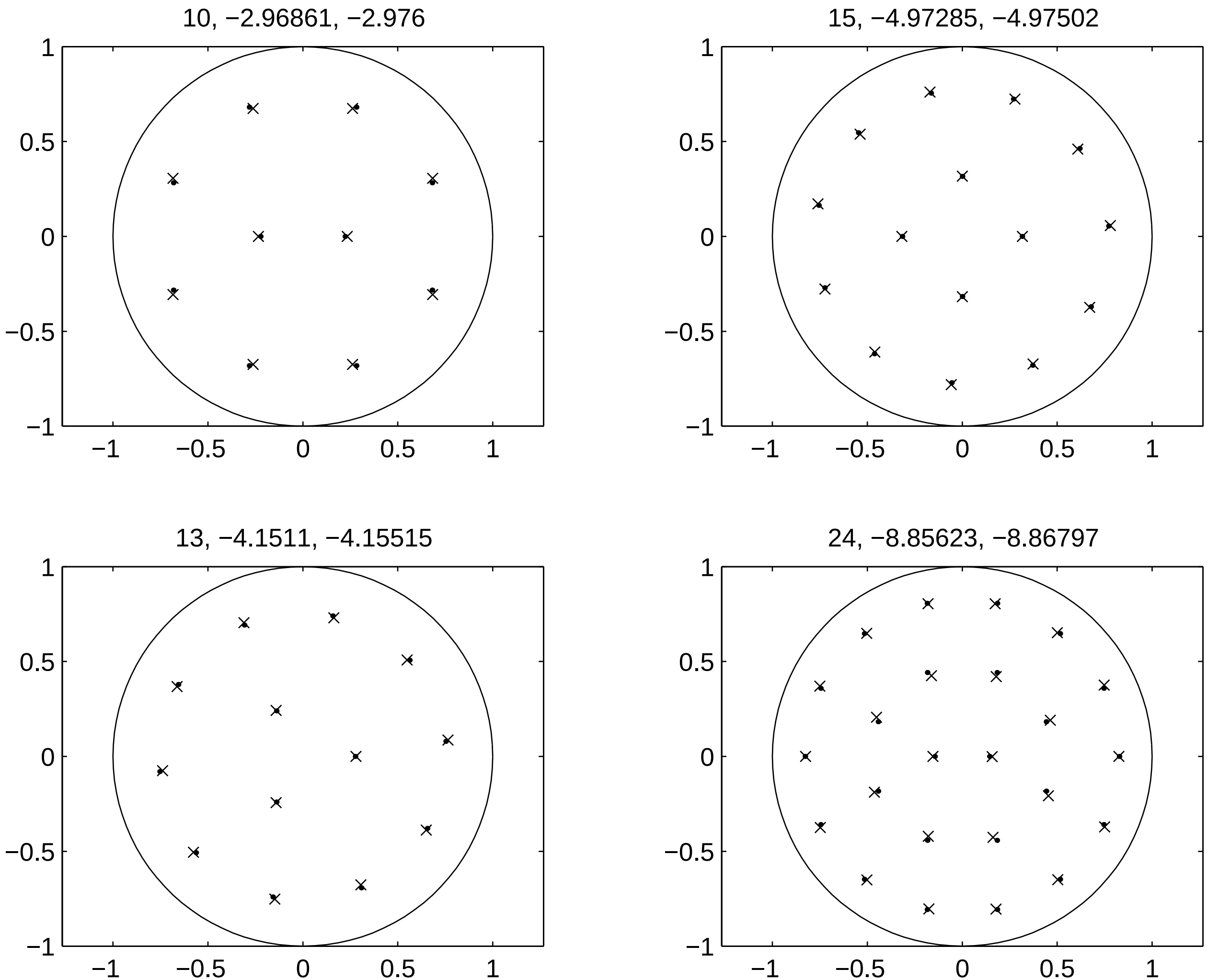
Constrained optimization on up to 3 rings



Full optimization of K traps



Comparison


10, −2.96861, −2.976
15, −4.97285, −4.97502
13, −4.1511, −4.15515
24, −8.85623, −8.86797
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1

Conclusion

- We looked at three very different problems: vortex dynamics; spike dynamics and first mean-passage time
- All three problems reduce to ***nonlocal particle aggregation model*** with Newtonial repulsion
- In the limit of large number of particles, the steady state approaches a ***uniform distribution.***
- Spectral equivalence of aggregation and vortex model shows stability

These papers are available for download from my website: `http://www.mathstat.dal.ca/~tkolokol`

Thank you! Any questions?