Curved interfaces in perturbed Allen-Cahn system

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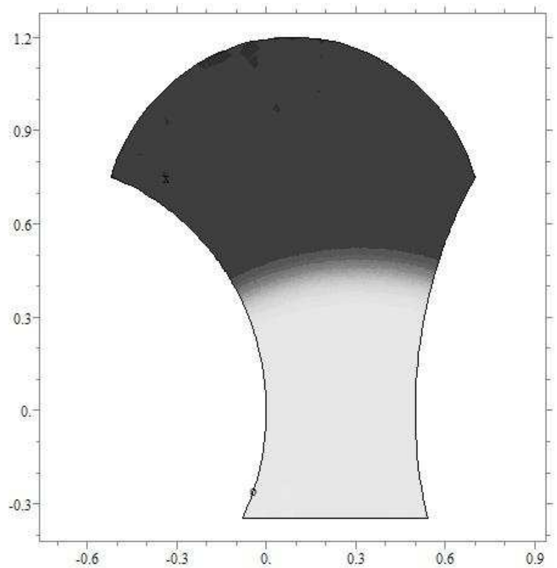

Joint work with

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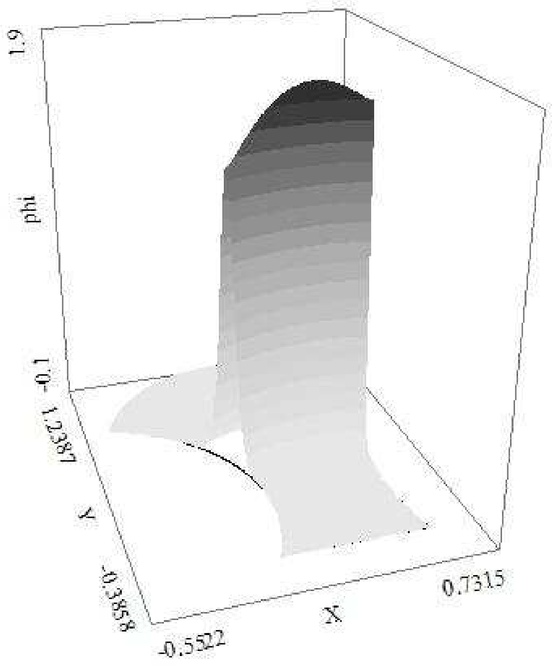


Introduction

We consider the perturbed Allen-Cahn equation:

$$\begin{cases} u_t = \varepsilon^2 \Delta u + f(u) + \varepsilon g(u), & x \in \Omega \subset \mathbb{R}^2\,, \\ \partial_n u = 0, & x \in \partial\Omega\,. \end{cases} \qquad \text{(PAC)}$$

Here, Ω is a smooth two-dimensional domain and $f(u)$ is a smooth function having the following properties:

- f has three roots $u_- < u_0 < u_+$ with $f'(u_\pm) < 0$
- $\int_{u_-}^{u_+} f(u)\, du = 0$

and $g(u)$ is any smooth function function with $\int_{u_-}^{u_+} g(u)\, du \neq 0$.

Some known results

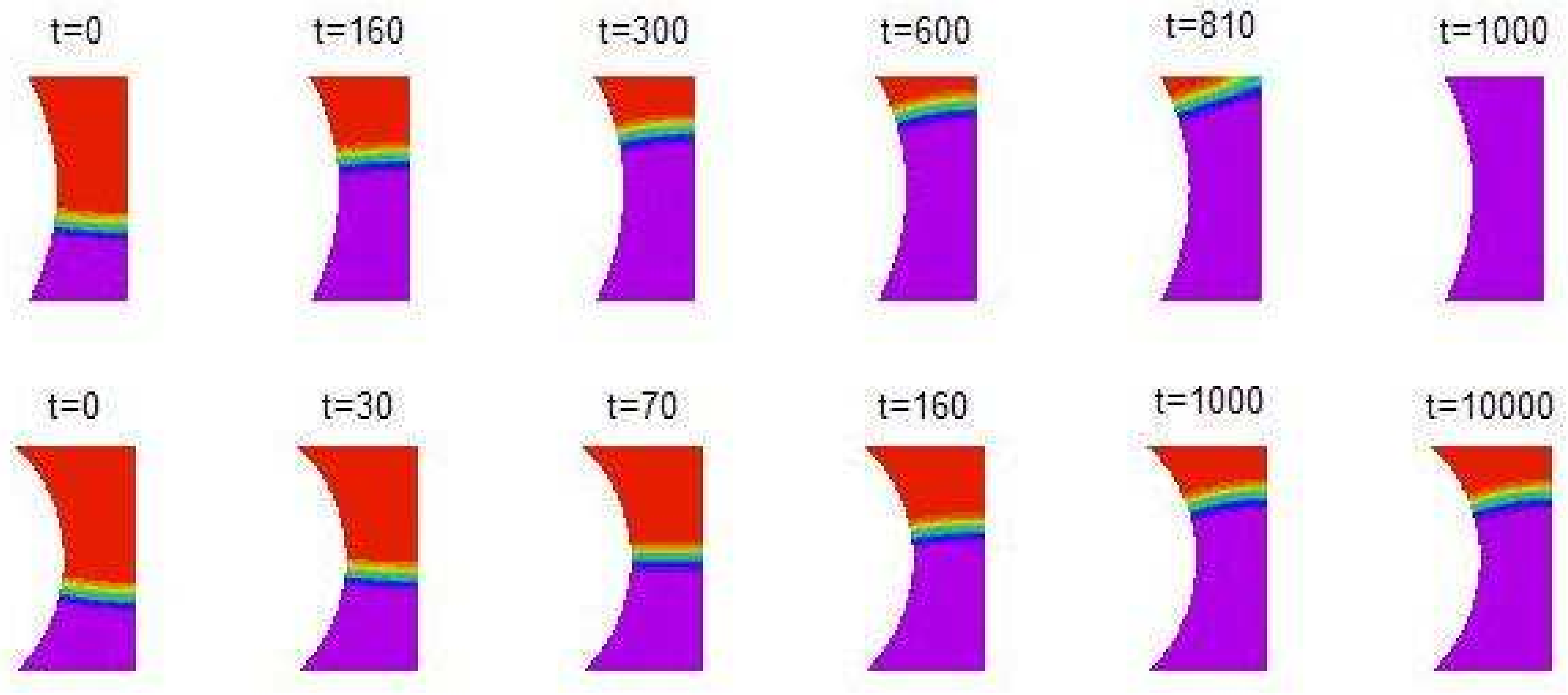
Standard form:

$$\begin{cases} u_t = \varepsilon^2 \Delta u - 2(u - \varepsilon a)(u^2 - 1), & x \in \Omega \subset \mathbb{R}^2 \\ \partial_n u = 0, & x \in \partial\Omega \end{cases}.$$

Standard A-C corresponds to $a = 0:$

- In 1-D, the steady state is given by $u = \pm \tanh(x/\varepsilon)$.
- In 2-D, the profile is 1-dimensional in some direction; the zero set $u = 0$ is a straight line, intersects boundary transversally.
- Such straight interface is stable (unstable) provided it is a local min (max) of the distance function. [Kowalczyk, 05]
- Time dependent solution evolves by mean curvature law until the interface merges with the boundary or becomes straight. [RSK, 89]

Effect of perturbation: numerics



Effect of perturbation: asymptotics

Let $U_0(z)$ be a solution to

$$U_0''(z) + f(U_0) = 0, \quad U \to u_\pm \text{ as } z \to \pm\infty.$$

and define

$$\hat{R} = -\frac{\int_{-\infty}^{\infty} U_0'^2(z)dz}{\int_{u_-}^{u_+} g(u)du} \quad (1)$$

Suppose that there exists a circle of radius $\hat{R}$ which intersects $\partial\Omega$ orthogonally, and let p be its center. Then in the limit $\varepsilon \to 0$ we have

$$u(x) \sim U_0\left(\frac{\hat{R} - |p - x|}{\varepsilon}\right), \quad \varepsilon \to 0 \quad (2)$$

any solution to (PAC) of the form (2) must satisfy (1).

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Derivation for cone-shaped domain

- Solution is radially symmetric:

$$\varepsilon^2 u_{rr} + \frac{1}{r} u_r + f(u) + \varepsilon g(u) = 0$$

- Solvability condition determines radius

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Main stability result

Consider an interface at an equilibrium whose radius is $\hat{R}$. Let ℓ be its length let κ_+, κ_- be the curvatures of the boundary at the points which intersect the interface. Consider the stability problem associated with (PAC),

$$\begin{cases} \lambda\phi = \varepsilon^2 \Delta\phi + f'(u)\phi + \varepsilon g'(u)\phi, & x \in \Omega \\ \partial_n \phi = 0, & x \in \partial\Omega. \end{cases} \tag{EP}$$

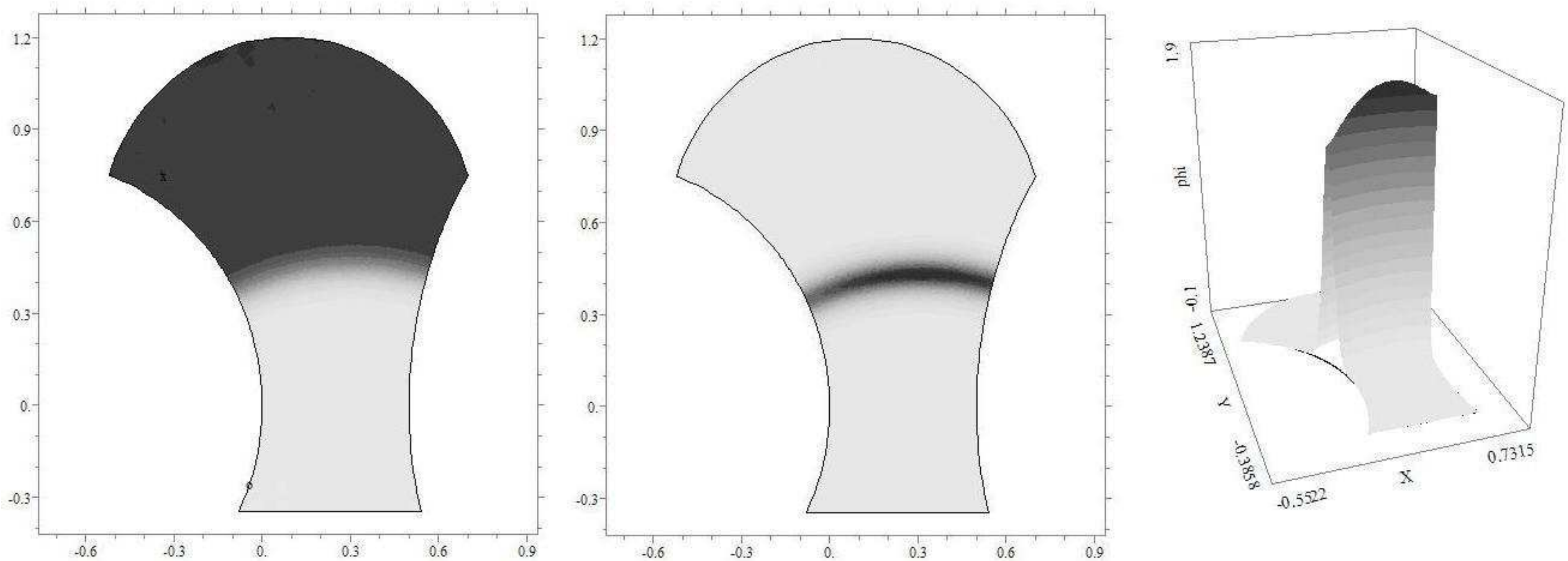
In the limit $\varepsilon \to 0$, we have $\lambda = \varepsilon^2 \lambda_0$ where λ_0 satisfies

$$\lambda_0 = \frac{1}{\hat{R}} - \mu^2 \quad \text{where} \quad \tan(\ell\mu) = -\frac{\mu(\kappa_+ + \kappa_-)}{\mu^2 - \kappa_+\kappa_-} \tag{3}$$

or

$$\arctan\left(\frac{-\kappa_+}{\mu}\right) + \arctan\left(\frac{-\kappa_+}{\mu}\right) = \ell\mu. \tag{4}$$

Example



(a) (b) (c)

- $u_t = \varepsilon^2 \Delta u - 2(u - \varepsilon a)(u-1)(u+1); \quad a = 0.55, \quad \varepsilon = 0.06;$ $\kappa_- = -1.25, \quad \kappa_+ = -0.667, \quad l = 0.6486.$
- $\hat{R}_{\text{theory}} = 1/(2a) = 0.9091; \ \hat{R}_{\text{numerical}} = 0.9066$
- $\lambda_{\text{theory}} = 0.00506. \ \lambda_{\text{numerical}} = 0.00504.$

Instability on a cone domain

Radially symmetric case:

- Expand

$$\phi = \Phi_0(z) + \varepsilon \Phi_0 + \cdots$$

$$r = \hat{R} + \varepsilon z$$

- Apply solvability condition
- End result:

$$\lambda_0 = \frac{1}{\hat{R}^2} > 0.$$

- Interface is unstable on a cone [or any convex domain]

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Geometric eigenvalue problem

... Equivalently, $\lambda = \varepsilon^2 \lambda_0$ where λ_0 satisfies

$$\begin{cases} w'' + (\lambda_0 - \hat{R}^{-2})w = 0 \\ w'(-\ell/2) + \kappa_- w(-\ell/2) = 0 \\ w'(\ell/2) + \kappa_+ w(\ell/2) = 0. \end{cases} \qquad \text{(GEP)}$$

- The standard AC model corresponds straight interface, $\hat{R}^{-1} = 0$. In this case (GEP) is the same as the formula derived by Kowalczyk (2005).
- In the case $\hat{R}^{-1} = 0$, stability threshold $\lambda_0 = 0$ occurs when $l + \kappa_+^{-1} + \kappa_+^{-1} = 0$. Geometrically, the circles tangent to the boundaries are concentric.
- QUESTION: What is the threshold in general case $(\hat{R}^{-1} \neq 0)$?

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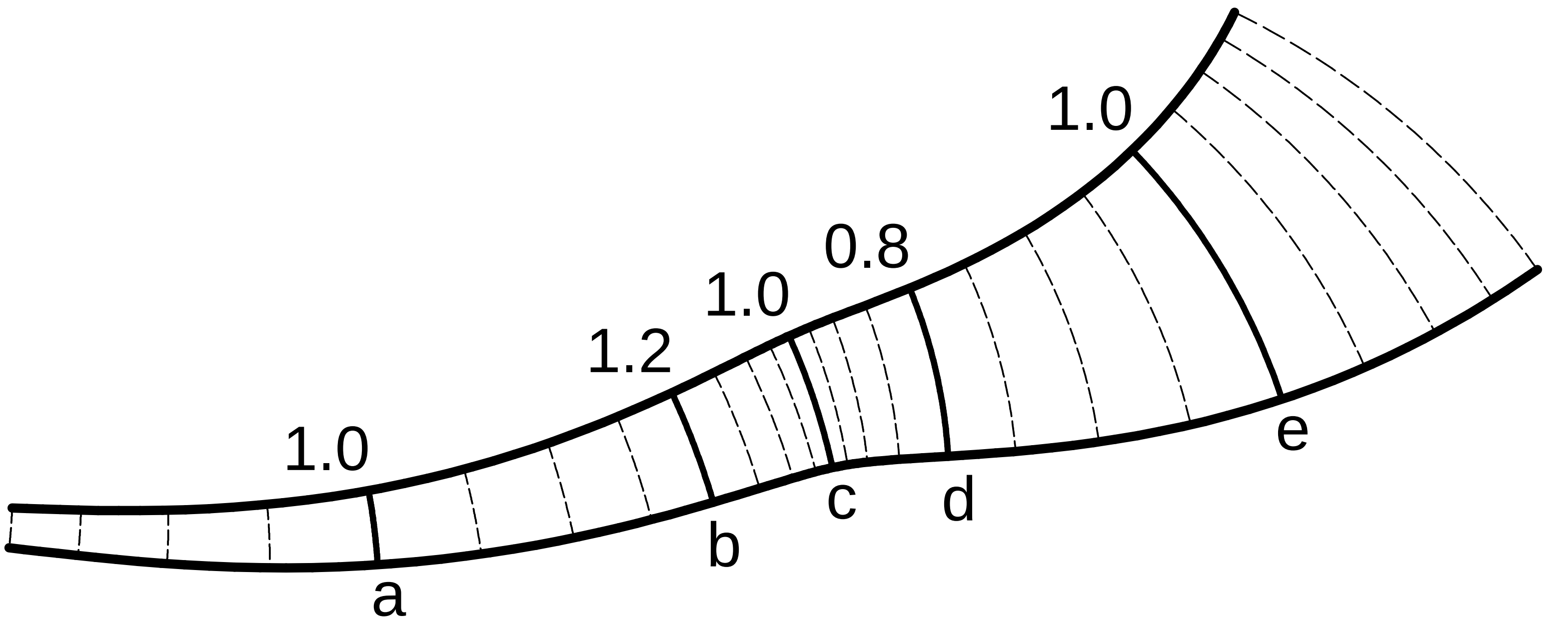
Geometric criterion for stability



Stability $\iff R'(s) < 0$ whenever $R = \hat{R}$

Example: If $\hat{R} = 1$ then curve c represents the location of a stable interface, whereas curves a and e correspond to unstable interfaces.

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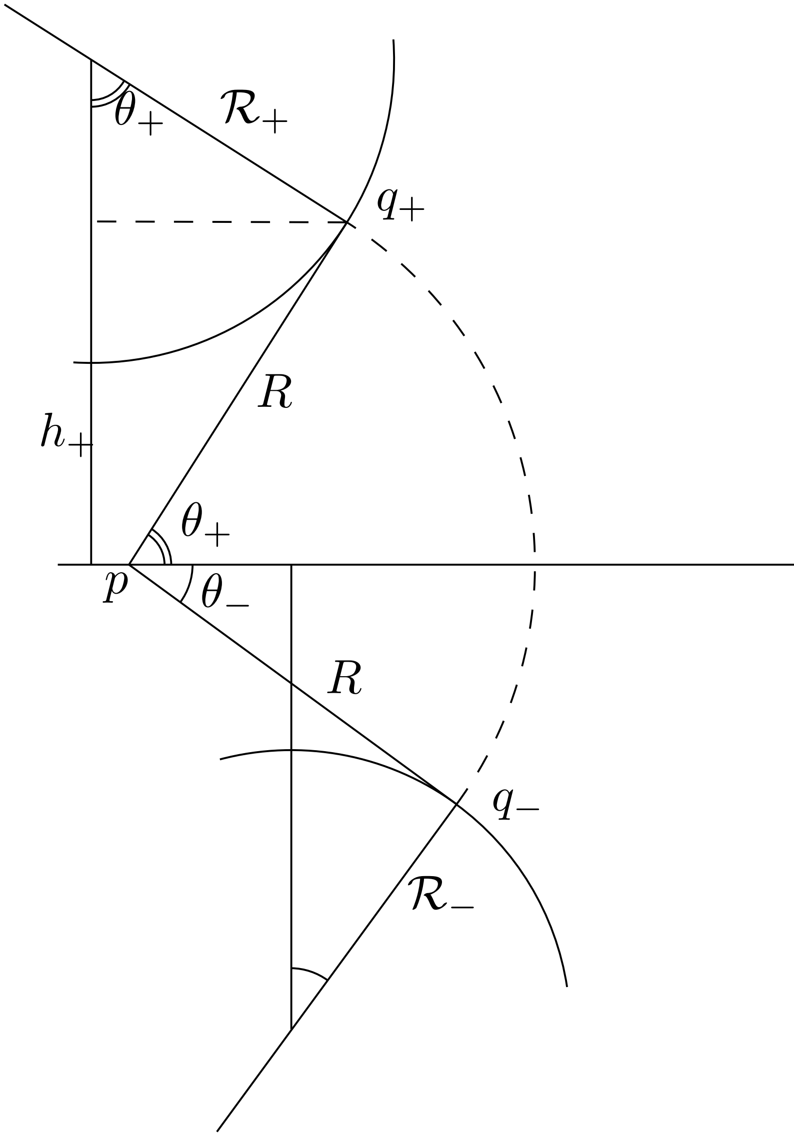
Derivation of Geometric Criterion

- $R = \frac{\mathcal{R}_+(1-\cos\theta_+)+h_+}{\sin\theta_+}$.
- $R' = 0 \iff \arctan(\frac{R}{\mathcal{R}_\pm}) = \theta_\pm$
- $\theta_+ = \ell_+/R,\ \theta_- = \ell_-/R$ and $\ell = \ell_+ + \ell_-$
- $R' = \iff \lambda_0 = 0$
- For a cone domain, $\lambda_0 = 1/\hat{R}^2 > 0$.
- By continuity, $\lambda_0 > 0$ whenever $R' > 0$.



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Open question: Stability of tractrix

- In general, we have

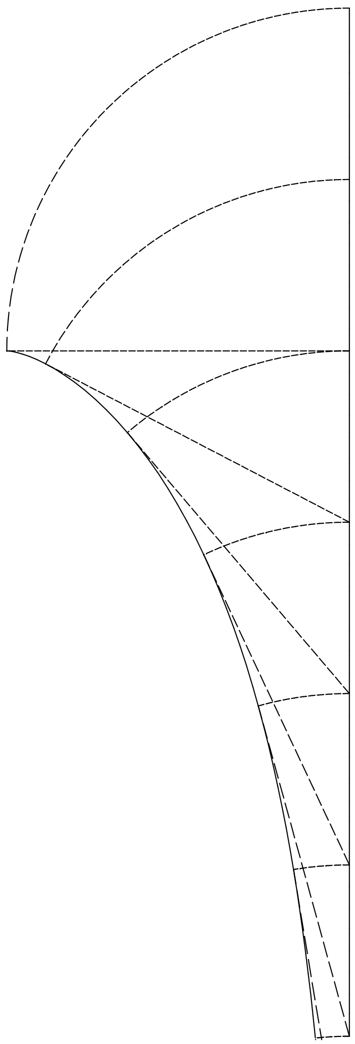
$$R\frac{d\theta}{ds} = p_1' \sin\theta - p_2' \cos\theta.$$

where p, θ, R are functions of s, and the boundary of the domain is given by $p + R(\cos\theta, \sin\theta)$.

- If R is constant then every circle that intersects the boundary orthogonally has the same radius. The resulting curve is a *tractrix* given by

$$x = \hat{R}(-t + \tanh(t)), \quad y = \hat{R}\,\mathrm{sech}(t).$$

- $\lambda_0 = 0, \lambda = O(\varepsilon^3)$? Open question: Determine stability.



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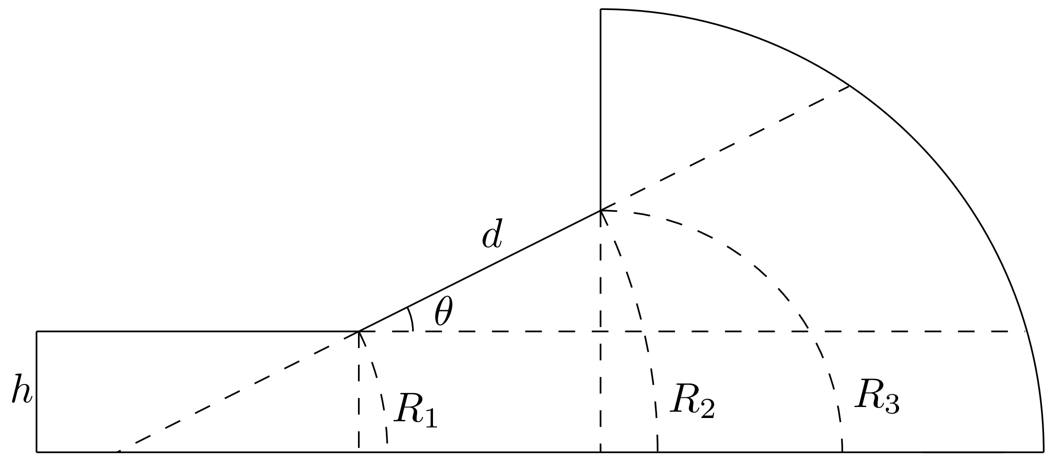
Open question: Corner junctions

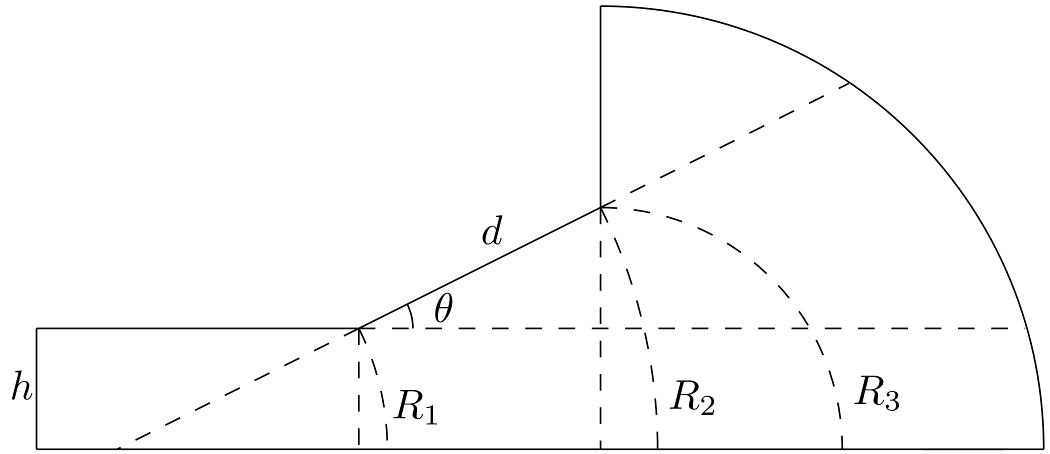


$$R_1 = h/\sin\theta, \quad R_2 = R_1 + d, \quad R_3 = h + d\sin\theta$$

If we “smooth out” the corners and start with an interface at the left...

- If $\hat{R} > R_1$ then interface stops at left corner
- If $R_1 > \hat{R} > R_3$ then interface stops at right corner
- If $\hat{R} < \min(R_1, R_3)$ then interface propagates and dies.

Open question: Corner junctions



Example: $h = 0.2, d = 0.5, \theta = \pi/6$; then

$$R_1 = 0.6, R_2 = 0.9, R_3 = 0.45$$

- Take $\hat{R} = 0.65$; interface stops at first corner
- Take $\hat{R} = 0.5$; interface stops at second corner
- Take $\hat{R} = 0.4$; interface goes through
- Open question: compute λ_0 for corner interface...