Random Walks and the Advection and Diffusion of Uncertainty

Victor Bomers

Contents

List of Figures

1 Introduction

This paper presents a brief overview of the study of random walks and, more generally, the evolution of uncertainty over time.

The study of the random displacement of particles was first documented the 19th century by the botanist Robert Brown (Gardiner, 1983). In 1827, Brown observed that small grains of pollen, when suspended in water, moved in animated and irregular ways. Brown initially believed that the movement of the pollen particles was due to some sort of life force that was associated with the grains but he was forced to dismiss that hypothesis after observing that other non-living particles, such as glass, behaved in a similar way. Over his lifetime, Brown was unable to provide an explanation for the phenomenon that he was the first to document, but because of his initial work on the subject the random displacement of a particle is sometimes referred to as Brownian Motion.

In 1905, the physicist Albert Einstein was the first person to explain Brownian Motion when he suggested in his paper *On the Motion, Required by the Kinetic Theory of Heat, of Small Particles Suspended in a Stationary Liquid* that the apparently random motion of particles suspended in a liquid was caused by collisions between the particles and the molecules that made up the liquid. What is interesting about Einstein's explanation is that though he approached the problem of Brownian Motion as a physicist, he recommended that the the system underlying the collisions was so complicated that it should only be studied probabilistically and explained statistically (Gardiner, 1983).

This paper does not discuss Brownian Motion directly but rather focuses on the statistical characteristics of random walks. Over the course of this paper the subject of random walks is treated first in discrete time and discrete space, then in discrete time and continuous space and finally in continuous time and continuous space.

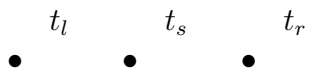
	discrete space	continuous space
discrete time	Section 2	Section 4
continuous time		Section 5

Section 2 introduces a discrete time, discrete space random walk in one dimension and discusses its evolution over time if it is unbounded. Section 3 discusses the

evolution of a random walk in discrete space in more detail and provides examples, in one and two dimensions, of what happens to the evolution of a walk if it is spatially bounded. Section 4 marks the shift in the context of the discussion of random walks from discrete space to continuous space and introduces stochastic systems which are systems that have both a deterministic and a random component. In this section the Self Advecting Vortex Model (SAVM) model for ocean circulation is used as an example to illustrate the evolution of stochastic systems over time. Finally, Section 5 provides a brief discussion of evolution of stochastic systems in continuous time and continuous space and refers to the Fokker-Plank Equation.

2 A Random Walk on a One Dimensional Unbounded Lattice

Consider a discrete time, discrete space random walk in one dimension, where at every time step, a particle has an equal probability of staying it in its current state or moving instantaneously one state to the left or to the right. At any time step, the probability of a particle moving to a neighbouring state can be represented with the following diagram



where the center point indicates the position of the particle at time n and t_s, t_l and t_r represent the transition probabilities of the particle staying in the same state or moving one state to the left or to the right respectively. It should be noted that t_s, t_l and t_r must satisfy the the condition that $t_r + t_l = 1 - t_s$ and that for this random walk $t_s = t_l = t_r = 1/3$.

The displacement between the states of such a random walk can be described by the stochastic process D_m, where D indicates the set of displacements of the particle between times $m-1$ and m for $m = 1, \ldots,$ n. Since, in this random walk, at any time step, the particle is only able to stay in its current state or move one state to the right or to the left, each with an equal probability of 1/3, the stochastic process D_m can be summarized as follows:

d_i	-1	0	1
$p(d_i)$	1/3	1/3	1/3

For this random walk it is assumed that the displacement of the particle between any two time successive steps is independent of the displacement of the particle between any other two successive time steps. The expected value and variance of the displacement of the particle are

$$\begin{aligned} E[D] \equiv \mu_d &= \sum_{i=1}^{3} d_i p(d_i) && (1) \\ &= -1(1/3) + 0(1/3) + 1(1/3) && (2) \\ &= 0 && (3) \end{aligned}$$

$$
\begin{aligned}
Var[D] \equiv \sigma_d^2 &= E[(D-\mu_d)^2] = E[D^2] = \sum_{i=1}^{3} d_i^2 p(d_i) && (4)\\
&= -1^2(1/3) + 0^2(1/3) + 1^2(1/3) && (5)\\
&= 2/3. && (6)
\end{aligned}
$$

The total displacement of the particle between at time n be determined by summing the displacements of the particle between time 0 and time n and this information can be combined with the information about the initial state of the particle to determine the state of the particle at time n. The position of the particle at time n can therefore be expressed as $S_n = S_0 + \sum_{m=1}^{n} D_n$, where S_0 indicates the initial state of the particle. It follows then that the state of the particle at time n can be described by the stochastic process S_n, where S denotes the set of states that the particle can inhabit and n denotes time. It is assumed that the random walk of the particle is unbounded and that the number of states of the walk increases with time such that at any time n there are a total of $K = 2n + 1$ states and $S = \{-\lfloor K/2 \rfloor, \ldots, 0, \ldots, \lfloor K/2 \rfloor\}$.

For this random walk it is also assumed that the one dimensional displacement of the particle can be mapped on to the x-axis such that the particle is always initially located at the origin, so $S_0 = 0$, and the equation that describes the state of the particle at time n can simplified to $S_n = \sum_{m=1}^{n} D_m$ so that S_n indicates the position of the particle on the x-axis. In general, the state of the particle at time n depends only on the state of the particle at time $n-1$ and on the displacement of the particle over the interval m, which is the interval between time $n-1$ and time n, so S_n can also be expressed as $S_n = S_{n-1} + D_m$. Since $S_{n-1} = \sum_{m=1}^{n-1} D_m$, expressing S_n as the sum of the state of the particle at time $n-1$ and the displacement of the particle over the interval m is equivalent to expressing S_n as the sum of the displacements of the particle up to time n.

The expected value and variance of the state of the particle therefore

$$
\begin{aligned}
E[S] \equiv \mu_s &= E[\sum_{m=1}^{n} D_m] && (7)\\
&= E[D_1] + E[D_2] + \cdots + E[D_n] && (8)\\
&= \mu_d + \mu_d + \cdots + \mu_d && (9)\\
&= n\mu_d = 0 && (10)
\end{aligned}
$$

$$
\begin{aligned}
Var[S] \equiv \sigma_s^2 &= Var[\sum_{m=1}^{n} D_m] && (11)\\
&= Var[D_1] + Var[D_2] + \cdots + Var[D_n] && (12)\\
&= \sigma_d^2 + \sigma_d^2 + \cdots + \sigma_d^2 && (13)\\
&= n\sigma_d^2 = \frac{2}{3}n. && (14)
\end{aligned}
$$

In the five realizations of the random walk, shown in the plot on the left in Figure 1, it can be seen that the ensemble of the realizations has a mean of 0 and that the variance of the ensemble increases with time. The statistical characteristics of the realizations shown on the left are summarized by the plots shown on right, which indicate the probability mass function, $p_n = p(S = \mathrm{s}, n)$, of the state of the random walk. The similarity between the shape of the probability mass function of the state of the random walk and a normal distribution is not coincidental because it can also be shown that the the probability mass function of the state of the random walk becomes approximately normally distributed as $n \to \infty$.

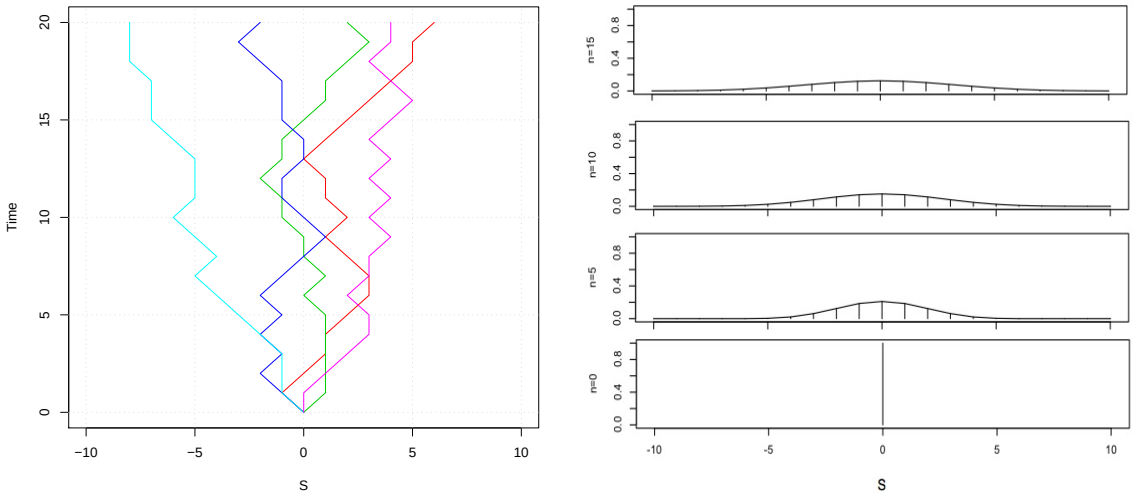


Figure 1: Five realizations of a random walk in one dimension.

Since D_m are iid with mean μ_d and variance σ_d^2 the Central Limit Theorem states

that the distribution of the normalized sum of $D_1,\ldots,D_n$ tends to the standard normal distribution as $n \to \infty$. This implies that the distribution of the state of the particle, S_n, normalized by a factor of σ_s, tends to the standard normal distribution as $n \to \infty$. That is:

$$\frac{D_1 + \cdots + D_n - n\mu_d}{\sigma_d\sqrt{n}} = \frac{\sum_{m=1}^{n} D_m - n\mu_d}{\sigma_d\sqrt{n}} = \frac{S_n - \mu_s}{\sigma_s}$$

so

$$P\left(\frac{S_n - \mu_s}{\sigma_s} \leq a\right) = P\left(\frac{S_n}{\sigma_s} \leq a\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-s^2/2} ds \text{ as } n \to \infty$$

It should be noted that the Central Limit Theorem applies to both continuous and discrete random variables. The only necessary condition for it to hold is that the random variables are iid. To understand the asymptotic convergence of the distribution of S_n, a discrete distribution, to the normal distribution, a continuous distribution, note that its variance, σ_s^2, increases with time (since $\sigma_s^2 = n \cdot 2/3$), so as n increases to infinity, the distance, ΔS_n, between the states S_n and S_{n+1}, relative to the width of distribution of S_n, decreases to zero. That is, as $n \to \infty$, $\Delta S_n \to 0$, so the distribution of S_n becomes asymptotically continuous.

The fact that S_n/σ_s is approximately normally distributed with mean 0 and variance 1 implies that S_n is approximately normally distributed with mean $\mu_s = 0$ and variance $\sigma_s^2 = n \cdot 2/3$. This is shown in Figure 2. In Figure 2, SSD is the the sum-of-squares of the difference between the probability mass function of S_n and the corresponding normal distribution it approximates. It should be noted that though the distribution of S_n is supposed to converge to a normal distribution asymptotically, convergence effectively occurs rather quickly in this case.

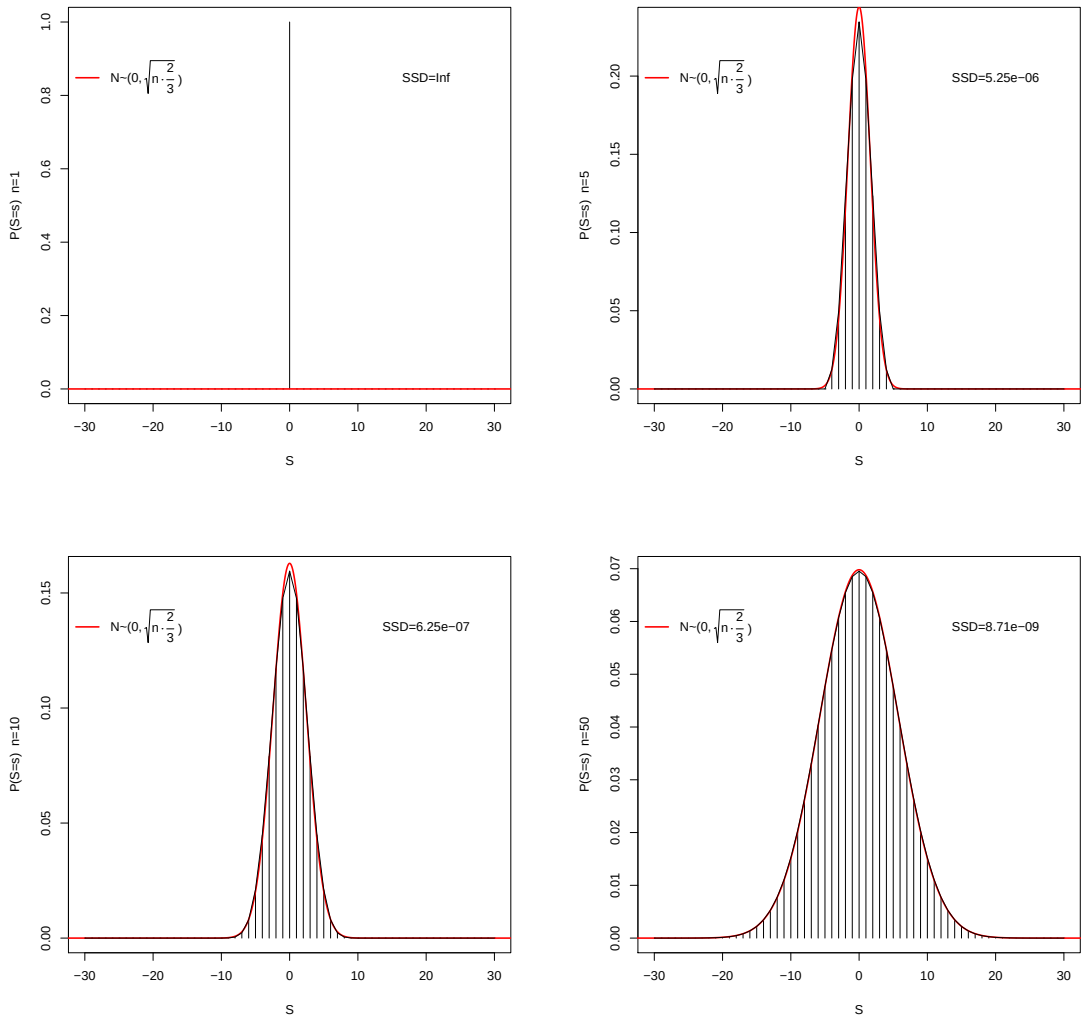


Figure 2: The evolution of the probability mass function of the state of the particle at time n and the corresponding normal distribution it approximates (indicated by the red line). It should be noted the the pmf is centered at 0 and that its variance increases by 2/3 units every time step as indicated by equations 10 and 14.

3 Generalized Random Walks on Bounded Lattices

Since the random walk introduced in Section 2 evolves according to the equation $S_n = S_{n-1} + D_n$, the probability mass function of S_n, p_n, evolves in a similar way. In fact, p_n, the probability of the particle being at any given state at time n depends only on p_{n-1}, the probability of the particle being in any given state at time $n-1$ and the transition probabilities of the particle moving from a state at time $n-1$ to a state at time n.

As such, the random walk satisfies the Markov property, which states that the probability of a particle being in a given state n at time n depends only on the probability mass function at time $n-1$ (Stewart, 1994). This is satisfied mathematically if

$$p\left(S_n = \mathrm{s}_n | S_0 = \mathrm{s}_0, \ldots, S_{n-1} = \mathrm{s}_{n-1}\right) = p\left(S_n = \mathrm{s}_n | S_{n-1} = \mathrm{s}_{n-1}\right).$$

This probability, which gives the conditional probability of a particle moving from state s_{n-1} to state s_n over one time step is known as a transition probability and it can be expressed in the general form

$$t_{ij} = p\left(S_n = j | S_{n-1} = i\right)$$

to indicate the probability of the particle moving from state i to state j. For the generalized random walks considered in this section let $k = 1, \ldots, K$ indicate the number of states in the random walk and let i, $i = -\lfloor K/2 \rfloor, \ldots, 0, \ldots, \lfloor K/2 \rfloor$, indicate the current state of particle in relation to the origin and j, $j = -\lfloor K/2 \rfloor, \ldots, 0, \ldots, \lfloor K/2 \rfloor$ indicate the subsequent state of the particle in relation to the origin. There are therefore K transition probabilities for each state, stacked in a column, in order to represent the probability of a particle in a given state moving to any of the other states, and for an K state system the transition probabilities are arranged into the KxK transition matrix $\mathbf{T}$. Each transition probability, $t_{i,j}$, must be a valid probability, and must therefore satisfy the condition that $0 \leq t_{i,j} \leq 1$, and in addition, the sum of each of the columns of the transition matrix must sum to 1.

Given the transition matrix $\mathbf{T}$, and $\mathbf{p_{n-1}}$, a vector that indicates the probability of a particle being in any given state at time $n-1$, the probability of a particle being at a given state at time n is given by the equation

$$\mathbf{p_n} = \mathbf{T}\mathbf{p_{n-1}},$$

which is a special case of the Chapman-Kolmogorov equation (Stewart, 1994), and the evolution of the random walk can be described by updating $\mathbf{p_n}$ at every time new time step. Alternatively, the probability of a particle being at a given state at time n can be found, given the initial location of the particle $\mathbf{p_0}$, using the equation

$$\mathbf{p_n} = \mathbf{T^n p_0}.$$

For the random walk described above, it is assumed that the initial location of the particle is known to be at the origin, with probability 1. This can be represented by the Kx1 probability mass vector, $\mathbf{p_n}$, whose elements $p(s_i), i = -\lfloor K/2 \rfloor, \ldots, 0, \ldots, \lfloor K/2 \rfloor$, indicate the probability of the particle being at state i at time n.

$$\mathbf{p_n} = \begin{bmatrix} 0_{-\lfloor K/2 \rfloor} \\ \vdots \\ \\ 0_{-1} \\ 1_0 \\ 0_1 \\ \vdots \\ 0_{\lfloor K/2 \rfloor} \end{bmatrix}$$

The evolution of the pmf of the random walk can therefore be updated by multiplying the KxK transition matrix $\mathbf{T}$ by the Kx1 $\mathbf{p_n}$. For an unbounded random walk, the number of states increases with n so the dimension of $\mathbf{T}$ and $\mathbf{p_n}$ also increase at every time step and the calculation of $\mathbf{p_n}$ can be difficult. In general, however, the transition matrix for an unbounded random walk is of the form

$$\mathbf{T} = \begin{array}{ccccc} \ddots & \vdots & & & \\ \ddots & t_l & 0 & 0 & \ldots \\ & t_s & t_l & 0 & \\ & t_r & t_s & t_l & \\ & 0 & t_r & t_s & \ddots \\ \ldots & 0 & 0 & t_r & \ddots \end{array}$$

The computational problem presented by an unbounded random walk can be overcome by augmenting the dimension of $\mathbf{T}$ and $\mathbf{p_n}$ at every time step, or by initially choosing the maximum number of steps of interest and constructing $\mathbf{T}$ and $\mathbf{p_n}$ accordingly. It should be noted, however, that these computations problem can also be avoided if $\mathbf{p_n}$ is calculated using convolution.

In most cases however, random walks are not unbounded and are in fact limited to a finite state space of size K. Thus, when a random walk is bounded, the asymptotic properties of the walk depend on the boundary conditions of the walk, specifically whether the particle is reflected or absorbed at the boundary.

If an random walk has "reflecting boundaries", particles are unable to move past certain states and if they are forced against the boundary they are bounced back to a state that they have already visited. For an example, consider the diagram below where the black circles represent accessible states and the vertical line represents the boundary. If there is a particle that is located at the leftmost black dot that tries to move to the state represented by the open circle, it will be bounced back by the boundary and end up at the black circle on the right.

For random walks with "reflecting boundaries", the boundary transition probability of a particle staying in its current state stays the same, but the probability of the particle moving away from the boundary is equal to the original probability of the particle moving away from the boundary plus the probability of the particle moving towards the boundary. The transition matrix for random walks with "reflecting boundaries" is therefore of the form

$$\mathbf{T} = \begin{bmatrix} t_s & t_l & \dots & \dots & 0 & \dots & \dots & 0 & 0 \\ t_r + t_l & t_s & & & \vdots & & & \vdots & \vdots \\ 0 & t_r & \ddots & & 0 & & & & \\ 0 & 0 & & \ddots & t_l & & & & \\ \vdots & \vdots & & & t_s & & & \vdots & \vdots \\ & & & & t_r & \ddots & & 0 & 0 \\ & & & & 0 & & \ddots & t_l & 0 \\ \vdots & \vdots & & & \vdots & & & t_s & t_l + t_r \\ 0 & 0 & \dots & \dots & 0 & \dots & \dots & t_r & t_s \end{bmatrix}.$$

As the number of steps of such random walks $\to \infty$ the probability of a particle being located a particular state becomes uniformly distributed over all of the states. This is show in Figure 3.

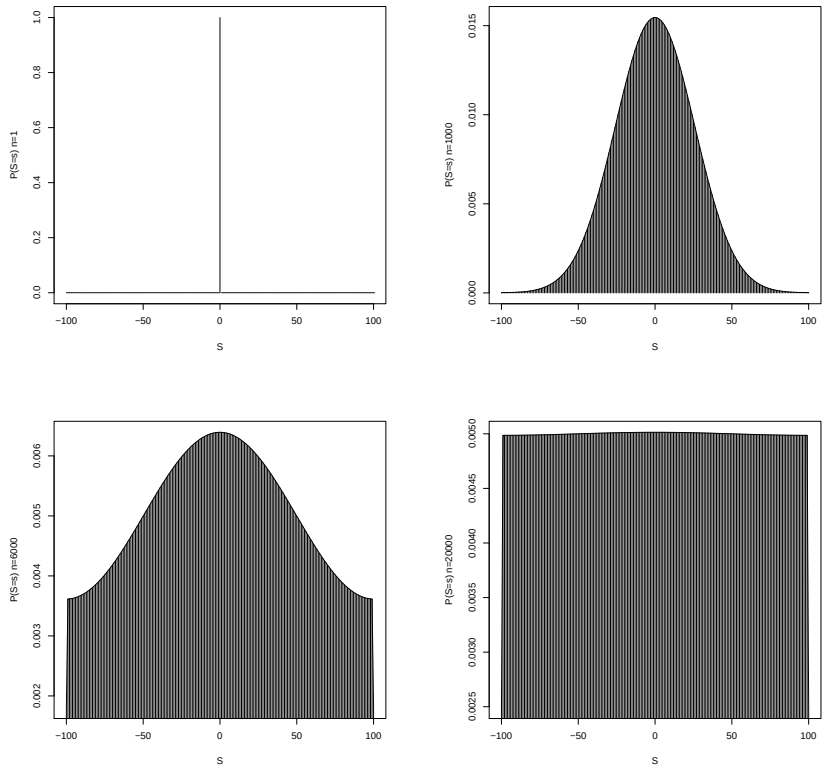


Figure 3: The evolution of the pmf of a random walk in one dimension with "reflecting boundaries".

If a random walk has "absorbing boundaries" any particles that reach the boundary get stuck there and do not move at subsequent time steps. This is indicated in the transition matrix of a random walk by setting the transition probabilities of points at the boundary equal to one, and setting the transition probability of a particle moving away from the boundary equal to zero. The transition matrix for a random

walk with "absorbing boundaries" is therefore of the form

$$\mathbf{T} = \begin{bmatrix} 1 & t_l & \dots & \dots & 0 & \dots & \dots & 0 & 0 \\ 0 & t_s & & & \vdots & & & \vdots & \vdots \\ 0 & t_r & \ddots & & 0 & & & & \\ 0 & 0 & & \ddots & t_l & & & & \\ \vdots & \vdots & & & t_s & & & \vdots & \vdots \\ & & & & t_r & \ddots & & 0 & 0 \\ & & & & 0 & & \ddots & t_l & 0 \\ \vdots & \vdots & & & \vdots & & & t_s & 0 \\ 0 & 0 & \dots & \dots & 0 & \dots & \dots & t_r & 1 \end{bmatrix}.$$

As $n \to \infty$ for random walks with "absorbing boundaries", all of the particles end up at the boundary points. For an unbiased walk, such as the walk presented in Section 2, all of the particles end up split between the two boundary points. This is shown in Figure 4. However, if a random walk has "absorbing boundaries" and is biased, the particles will be split at each boundary point proportionally to the bias of the walk.

For any type of random walk, it is also possible to have absorbing states that are not located at the boundaries of the state space. This can be achieved by changing some of the transition probabilities in the transition matrix, but ensuring that they satisfy the conditions mentioned above. If a random walk is bounded, and the propensity for a non-boundary absorbing state to absorb a particle is large enough, then any particles that follow the random walk will eventually come to be located at these absorbing states. An example of this is show in Figure 5.

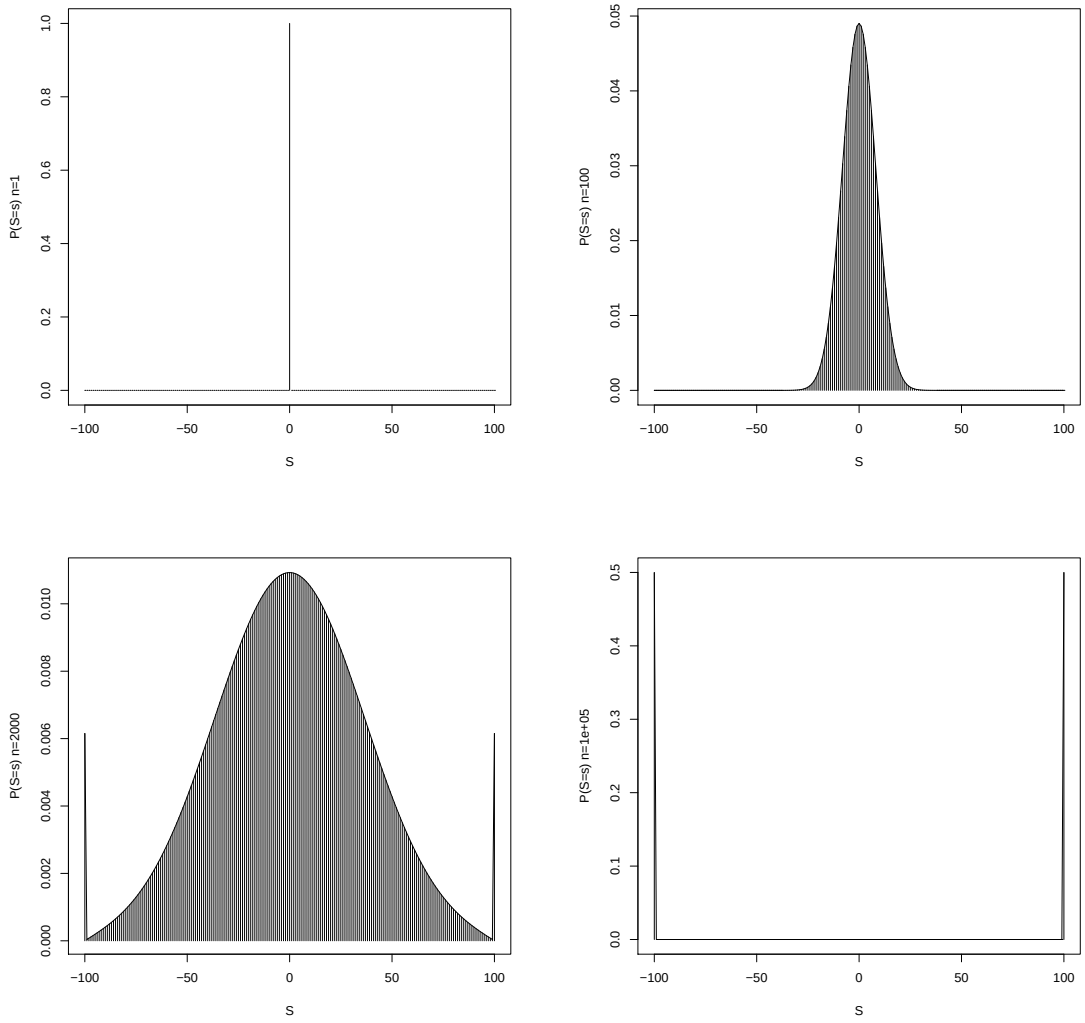


Figure 4: The evolution of the pmf of a random walk in one dimension with "absorbing boundaries".

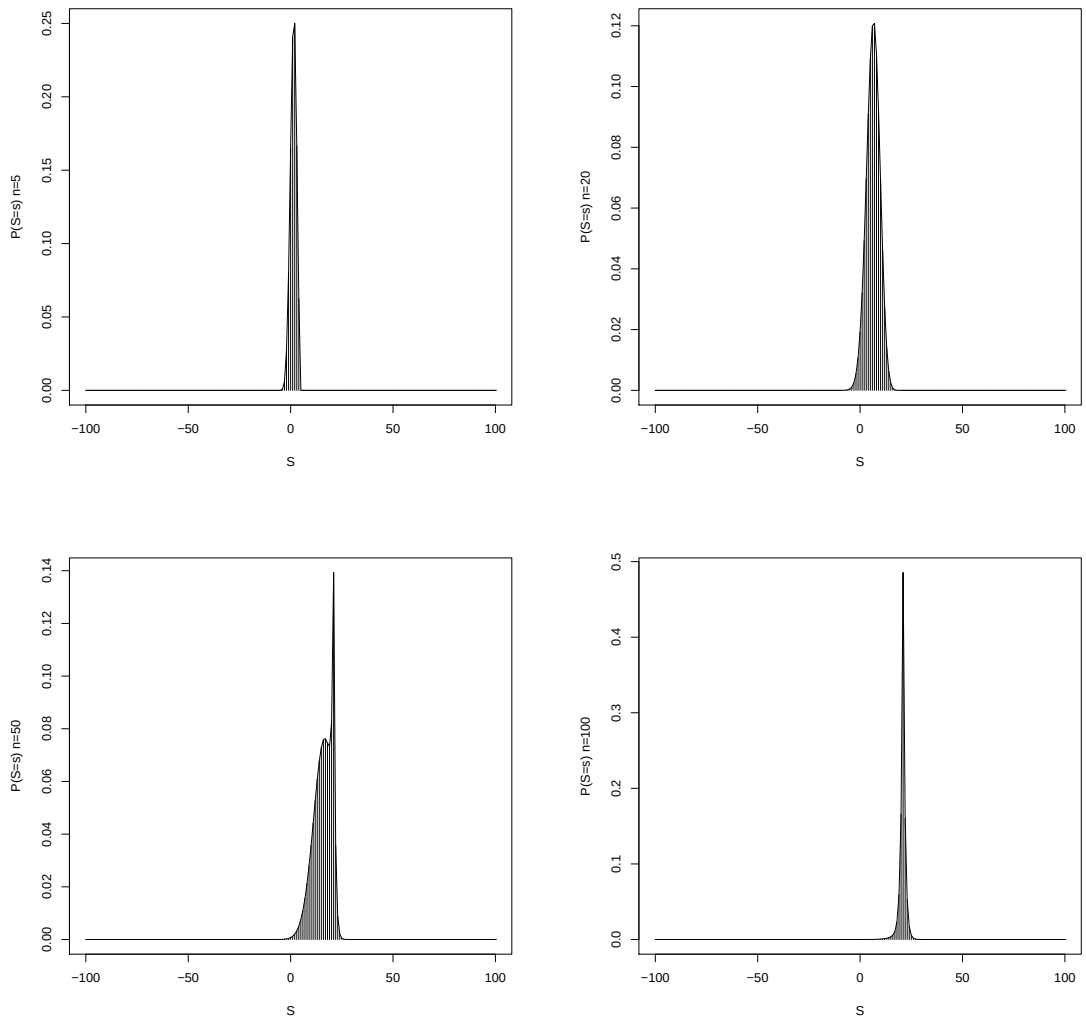
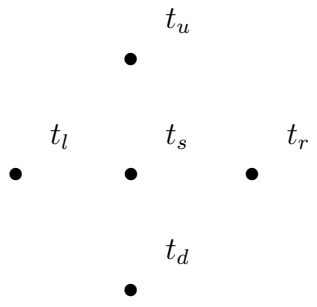


Figure 5: The evolution of the pmf of a random walk in one dimension with "reflecting boundaries" and an absorbing point to the right of the origin.

3.1 The Evolution of a Random Walk in Two Dimensions

Each of the results presented above can be generalized into 2 or more dimensions. To show this consider a discrete-time, discrete-space random walk in 2 dimensions where a particle has an equal probability of staying where it is or moving one step up, down, left or right at every time step. The probability of the particle moving to a neighbouring state can be represented with the following diagram



where the centre point indicates the position of the particle at time n and t_s, t_r, t_d, t_l, t_u represent the probability of the particle staying in its current state or moving to the corresponding neighbouring state. Note that $t_r + t_d + t_l + t_u = 1 - p_s$ and to begin let $t_s = t_r = t_d = t_l = t_u = 1/5$.

Initially, it is assumed that the position of the particle is known to be at the origina with probability 1. This can be represented by the K^2 x 1 state probability vector $\mathbf{p_n}$ which, indicates the probability of the particle being at state $s_{i,j}$. It should be noted that the state probability vector, $\mathbf{p_n}$ is formed by stacking the columns of the KxK state probability matrix $\mathbf{P_n}$, whose elements $p(s_{i,j}), i = -\lfloor K/2 \rfloor, \ldots, 0, \ldots, \lfloor K/2 \rfloor, j = -\lfloor K/2 \rfloor, \ldots, 0, \ldots, \lfloor K/2 \rfloor$, represent the probability of a particle being in any of the possible two dimensional states. Here i indexes the x coordinate of the current state and j indexes the y coordinate of the current state and i' and j' index the coordinates of the subsequent state.

$$\mathbf{P} = \begin{bmatrix} 0_{-\lfloor K/2 \rfloor,\lfloor K/2 \rfloor} & \cdots & 0_{0,\lfloor K/2 \rfloor} & \cdots & 0_{\lfloor K/2 \rfloor,\lfloor K/2 \rfloor} \\ & \ddots & \vdots & & \\ & & 0_{0,1} & & \\ \vdots & & 1_{0,0} & & \vdots \\ & & 0_{0,-1} & & \\ & & \vdots & \ddots & \\ 0_{-\lfloor K/2 \rfloor,-\lfloor K/2 \rfloor} & \cdots & 0_{0,-\lfloor K/2 \rfloor} & \cdots & 0_{\lfloor K/2 \rfloor,-\lfloor K/2 \rfloor} \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 0_{-\lfloor K/2 \rfloor,\lfloor K/2 \rfloor} \\ \vdots \\ 0_{-\lfloor K/2 \rfloor,-\lfloor K/2 \rfloor} \\ - \\ 0_{0,\lfloor K/2 \rfloor} \\ \vdots \\ 0_{0,1} \\ 1_{0,0} \\ 0_{0,-1} \\ \vdots \\ 0_{0,-\lfloor K/2 \rfloor} \\ - \\ 0_{\lfloor K/2 \rfloor,\lfloor K/2 \rfloor} \\ \vdots \\ 0_{\lfloor K/2 \rfloor,-\lfloor K/2 \rfloor} \end{bmatrix}$$

The evolution of the random walk can be described by the K^2xK^2 transition matrix $\mathbf{T}$, whose elements, $t_{i,j\to i',j'}$ contain the conditional probabilities of moving to state i', j' at time $n+1$ given that the particle is at state i, j at time n. As before, the elements of $\mathbf{T}$ must satisfy the following conditions

$$0 \leq t_{i,j\to i',j'} \leq 1 \qquad\qquad \sum_j t_{i,j\to i',j'} = 1.$$

If a two dimensional random walk has "reflecting boundaries", a particle that is located at the upper right state has a transitional probability of moving down one step that is equal to its normal transitional probability of moving down one step plus the normal transitional probability of moving up one step. Similarly, the transitional probability of the particle moving to the right would equal the normal transitional probability of moving right plus the normal transitional probability of moving left. Therefore, the transition matrix, in the simplest case where k=3, is as follows

$$\mathbf{T} = \left[\begin{array}{ccc|ccc|ccc} t_s & t_u & 0 & t_l & 0 & 0 & 0 & 0 & 0 \\ t_d + t_u & t_s & t_u + t_d & 0 & t_l & 0 & 0 & 0 & 0 \\ 0 & t_d & t_s & 0 & 0 & t_l & 0 & 0 & 0 \\ \hline t_r + t_l & 0 & 0 & t_s & t_u & 0 & t_l + t_r & 0 & 0 \\ 0 & t_r + t_l & 0 & t_d + t_u & t_s & t_u + t_d & 0 & t_l + t_r & 0 \\ 0 & 0 & t_r + t_l & 0 & t_d & t_s & 0 & 0 & t_l + t_r \\ \hline 0 & 0 & 0 & t_r & 0 & 0 & t_s & t_u & 0 \\ 0 & 0 & 0 & 0 & t_r & 0 & t_d + t_u & t_s & t_u + t_d \\ 0 & 0 & 0 & 0 & 0 & t_r & 0 & t_d & t_s \end{array}\right]$$

As in the one dimensional case, the probability of a particle being located at any given state become uniformly distributed over all the states as $n \to \infty$. An example of the evolution of the pmf for two dimensional walk with "reflecting boundaries" is show in Figure 6.

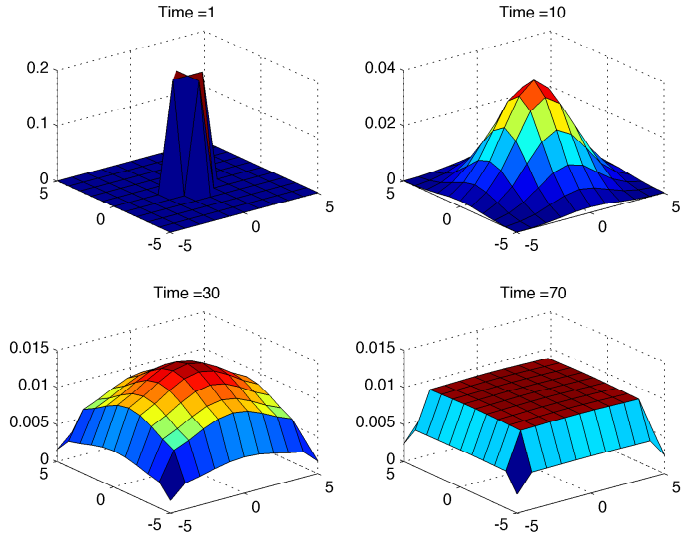


Figure 6: Evolution of the two dimensional pmf of a random walk with "reflecting boundaries".

If it is the case that a two dimensional walk has "absorbing boundaries" then the transitional probability of a particle staying at the boundary points is equal to

one, and the transitional probability of a particle moving away from the boundary is equal to zero. The transition matrix for such a walk would be of the form

$$\mathbf{T} = \left[\begin{array}{cccc|cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & t_l & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & t_l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & t_u & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_s & t_u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_d & t_s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_d & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t_r & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_r & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Again, as with a one dimensional random walk with "absorbing boundaries", as $n \to \infty$ the probability of a particle that follows a two dimensional walk with "absorbing boundaries" being in a certain state gets distributed among the boundary points, depending on the bias of the walk. An example of this evolution, for an unbiased random walk, is shown in Figure 7.

In general, the transition matrix of a two dimensional random walk can be expressed as an K^2 x K^2 partitioned block matrix

$$\mathbf{T} = \left[\begin{array}{ccccccccccc} \mathbf{N} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{RLB} & \mathbf{N} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{N} & \mathbf{L} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{N} & \mathbf{L} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & & & & \ddots & & & & & & \vdots \\ \vdots & & & & & \ddots & & & & & \vdots \\ \vdots & & & & & & \ddots & & & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{R} & \mathbf{N} & \mathbf{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{R} & \mathbf{N} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{N} & \mathbf{LRB} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{N} \end{array}\right]$$

where $\mathbf{N}, \mathbf{R}, \mathbf{L}, \mathbf{RLB}, \mathbf{LRB}$ and $\mathbf{0}$ are each KxK matrices. Here $\mathbf{N}$ is a tri-diagonal matrix that contains the 'normal' transition probabilities for a particle, $\mathbf{R}$ and $\mathbf{L}$

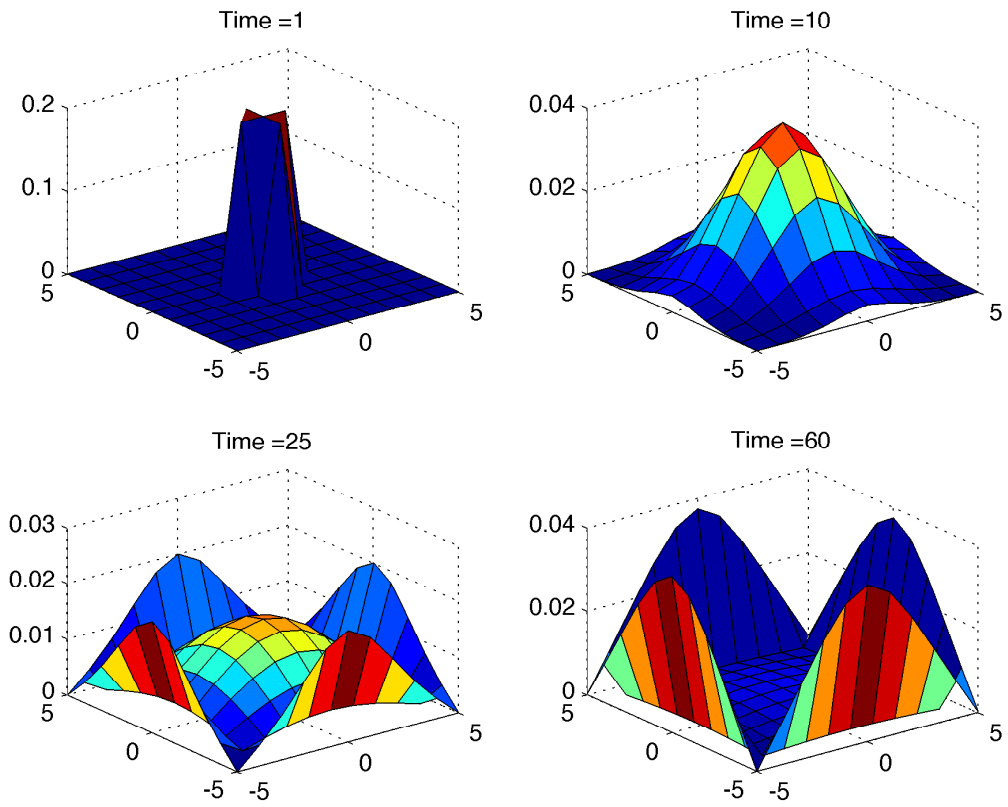


Figure 7: Evolution of the two dimensional pmf of a random walk with "absorbing boundaries".

are diagonal matrices that contain the transition probabilities of a particle moving right and left respectively, and **RLB** and **LRB** are diagonal matrices that contain the transition probabilities of a particle moving right and left, respectively, at the left and right boundaries. Figure 8 shows the evolution of the pmf of a random walk with "reflecting boundaries" that is biased to the right.

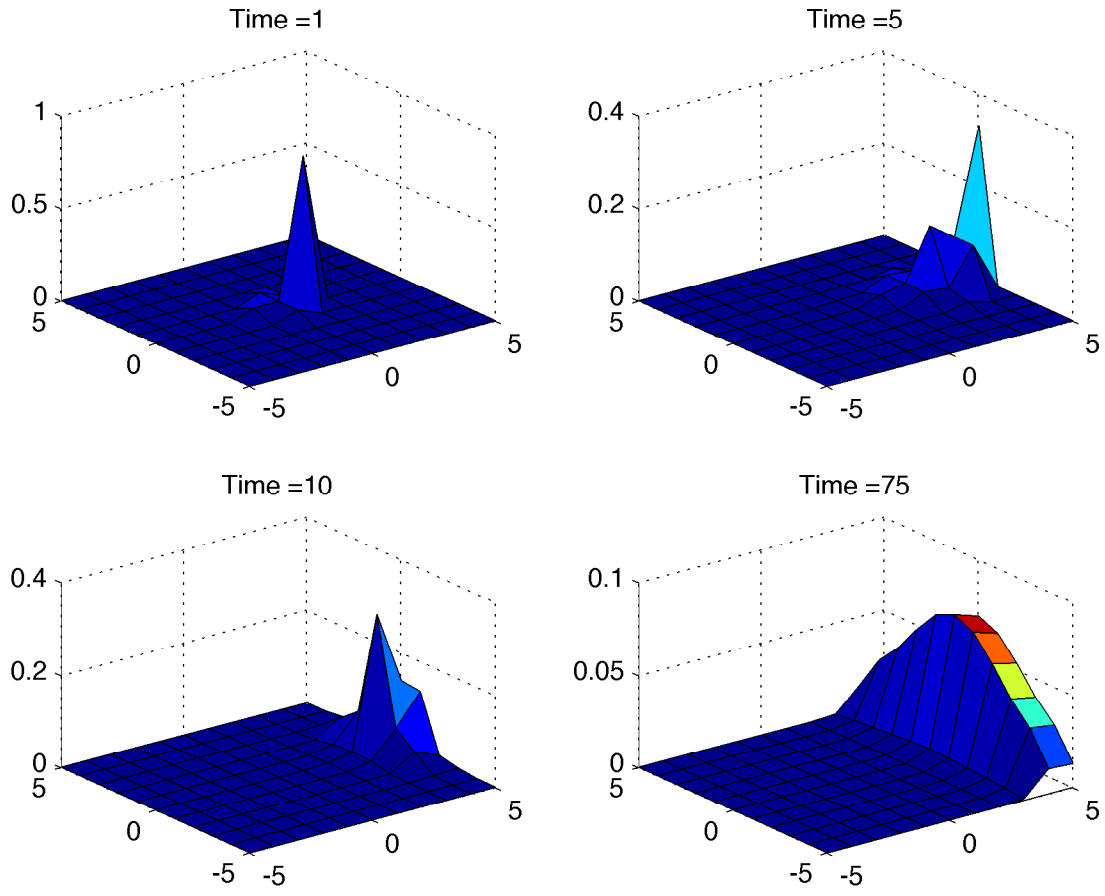


Figure 8: Evolution of the two dimensional pmf of a random walk with "reflecting boundaries" that is biased to the right

4 Discrete Time and Continuous Space: State Space Models

It was shown in Section 2 that the state of the random walk at time n could be expressed as

$$S_n = S_{n-1} + D_n.$$

This, however, is just a special case of a stochastic difference equation of the form

$$\mathbf{S}_n = f(\mathbf{S}_{n-1}) + g(\mathbf{S}_{n-1})\mathbf{D}_n$$

where f is a function that describes the deterministic displacement of the particle between time $n-1$ and time n and g is a function that determines the influence of the stochastic term, $\mathbf{D_n}$ on the displacement of the particle over the same interval. In general, both f and g can be linear or non-linear, $\mathbf{D_n}$ can be described by any type of distribution and it is assumed that n is discrete and $\mathbf{S}$ is continuous.

In the case that f and g are linear, and $\mathbf{D_n}$ is normally distributed, the probability density function of $\mathbf{S}$ at any time n is normally distributed because because it can be expressed as a linear combination $\mathbf{D}$ and any linear combination of normally distributed random variables is also normally distributed. Since the first two moments of the normal distribution, the mean and the variance, are sufficient statistics for the normal distribution, the statistical characteristics of the pdf of $\mathbf{S}$ and time n can be described completely by calculating its mean and variance from the previous $\mathbf{D_n}$.

In the alternate case, where one of either f or g are not linear, or $\mathbf{D_n}$ is not normally distributed, the probability density function of $\mathbf{S}$ at time n is much more difficult to compute. It cannot normally be summarized by one or two moments and instead has to be described by a pdf (Shumway, 2006). The Self Advecting Vortex Model presented in the next section provides an excellent example of what happens to the pdf of $\mathbf{S}$ in such a case.

4.1 The Self Advecting Vortex Model

The Self Advecting Vortex Model (SAVM) is a highly idealized model for ocean circulation that can be used to explain the advection and diffusion of particles in two-dimensional space. It is a special case of a Blinking Vortex Model (BVM), which is a classification for any type of model that involve vortices (generally two or

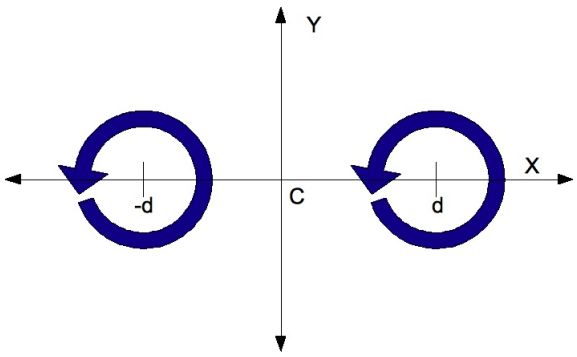


Figure 9: A diagram representing the Self Advecting Vortex Model (SAVM) considered in this paper. The centre of the two vortices are located at -d and d respectively. Assume that the point of reference rotates with the vortices so that the two vortices always appear as though they lie on the x-axis at a distance of 2d from one another.

more) that alternate between being on and off and hence 'blink'.

This paper considers a SAVM that consists of two vortices that rotate in a counter clockwise direction and are separated by a distance of 2d. A diagram that represents the SAVM used in this paper is shown in Figure 9. According to this model, each vortex spins alternately and as each vortex spins it creates a vector field in its surrounding environment that causes any particles in that environment to rotate around the vortex. A figure that shows the vector field that is created by the two vortices can be found in Appendix A. Over time this vector field develops distinct trajectories that determine the displacement of any surrounding particles. These trajectories are grouped together and outlined by special trajectories known as separatrices, which divide the groups of distinct trajectories from one another and act as boundaries between them. The presence of separatrices also indicates that the groups of trajectories are distinct, and that no groups of trajectories feed into each other. Figure 10 indicates the long run standing trajectories created by the SAVM used in this paper and Figure 11 indicates the separatrices that divide the different groups of trajectories.

Since the two vortices in the SAVM alternate between being on and between off, they are also affected by the vector field they create. That is, when a given vortex is off it is displaced according to the vector field that is created by the vortex that is on, and moves in a counter clockwise direction along a trajectory determined by the vector field. Over time, both vectors are affected by the vector field that they

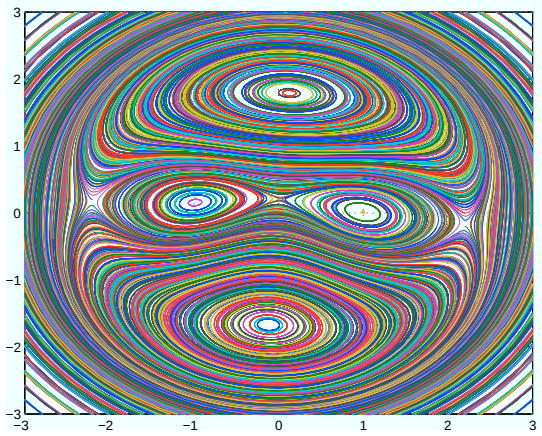


Figure 10: Long term trajectories induced in the surrounding environment by the two vortex SAVM considered in this paper. Note the localized loops near the initial vortices and the two ghost vortices, the uncrossed figure eight near the caused by the interaction of the two initial vortices, and the large circular trajectories near the edges of the grid.

create and rotate in a counter clockwise direction around a circle of radius d, centered at a given point C, such that the distance between the two vortices remains constant. For the sake of interpretation, assume that the point of reference rotates with the vortices so that the vortices can always be considered to lie along the x-axis.

At every time step, the location of the two vortices can be determined from the location of the centre point C, since the centre of the two vortices are located at a constant distance of d from C. If the centre point remains fixed then the general advection and diffusion of a particle can be described by the equation

$$\mathbf{s}_n = f(\mathbf{s}_{n-1}) + \mathbf{w}_n, \qquad \mathbf{w}_n \sim N(0, \boldsymbol{\sigma}_d^2)$$

where $\mathbf{S}_{n-1}$ denotes the location of the particle at time $n-1$, f is a non-linear function that describes the advection of the particle around C, and $\boldsymbol{\sigma}_d^2$ is known.

If f was a linear function the probability density function of $\mathbf{S}$ could be found at any time n by updating the mean and the variance of the random error term according to f. This is because the mean and variance are sufficient statistics for a normal distribution and the the probability density of $\mathbf{S}$ would be normally distributed since $\mathbf{w}$ is normally distributed and linear combinations of normally distributed random

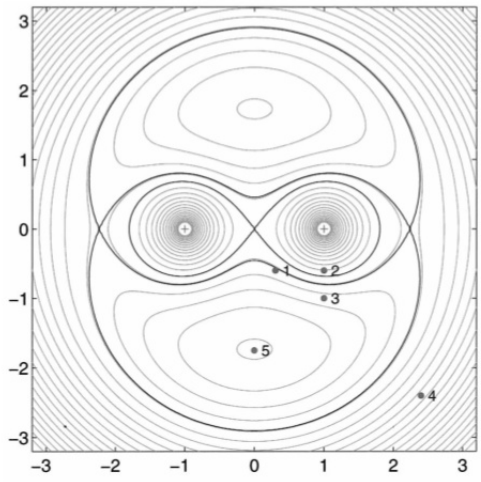


Figure 11: The separatrices for the two vortex SAVM model are shown above with bold lines. Note the location of the distinct trajectory regions (Kuznetsov et al. 2003).

variables are also normally distributed. However, because f is non-linear, the probability density function of $\mathbf{S}$ does not remain normally distributed over time and it must be described by a pdf at every time. Therefore, for non-linear stochastic systems such as the SAVM, the pdf of the system can quickly transform from a multivariate Gaussian distribution to a multivariate non-Gaussian distribution. This is shown in Figure 12. The red contour lines in the figure represent the 50th, 90th and 95th percentiles of the distribution, estimated by tracking an ensemble of particles (taken to approximate the pdf).

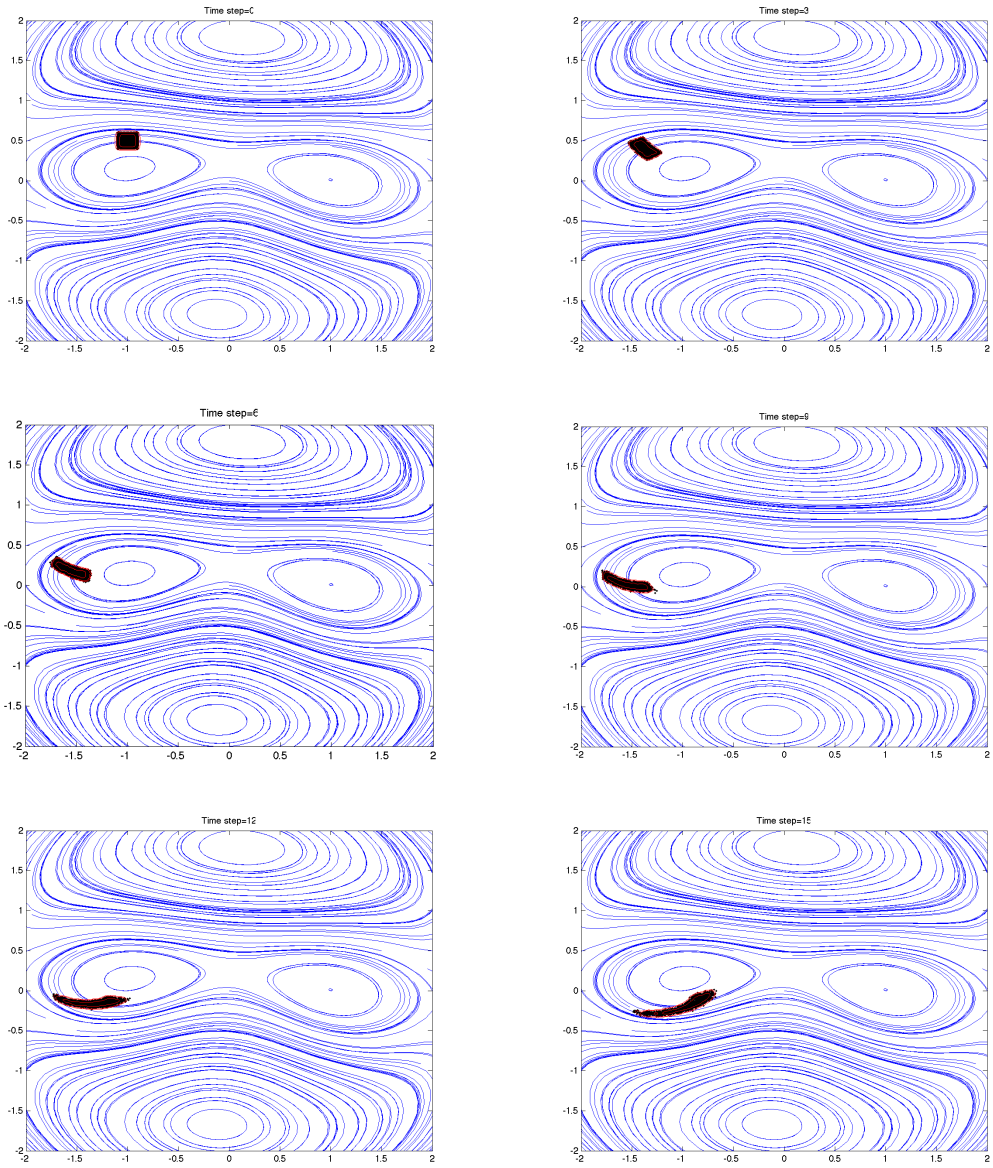


Figure 12: The evolution of a two dimensional pdf for the SAVM.

5 Continuous Space, Continuous Time: Stochastic Differential Equations

As the time between each time step tends to zero, the difference equations that are used to model the evolution of a stochastic system are replaced by differential equations. As a result, equations of the form $\mathbf{S}_n = f(\mathbf{S}_{n-1}) + g(\mathbf{S}_{n-1})\mathbf{D}_n$, that are discussed in Section 4, are replaced by equations of the form $\dot{\mathbf{s}} = f(\mathbf{s}) + g(\mathbf{s})\boldsymbol{\epsilon}(n)$, where the change in the state of the system over a given interval of time, $\dot{\mathbf{s}}$, can be expressed as linear combination of a deterministic function that describes the change in the state of the system, $f(\mathbf{s})$, and a stochastic white noise term, $\boldsymbol{\epsilon}$, weighted by $g(\mathbf{s})$. These stochastic differential equations relate the same information as stochastic difference equations except that they assume that both $\mathbf{S}$ and n are continuous.

The evolution of the probability density function of a one dimensional stochastic differential equation

$$\dot{s} = f(s) + g(s)\epsilon(n), \qquad \text{with initial conditions: s(0)=}z$$

can be described by the second order partial differential equation

$$\frac{\partial p}{\partial n} = -\frac{\partial}{\partial s}(f(s)p) + \frac{1}{2}\frac{\partial^2}{\partial s^2}(g^2(s)p)$$

which known as the Fokker-Plank Equation, where $p(s,n)$ is the probability density function of S and $p(s,0) = \delta(s-z)$ gives the initial conditions (Zwillinger, 1989). The solution of the Fokker-Plank equation is the probability density function of s and as a result the Fokker-Plank equation describes the continuous time, continuous space advection and diffusion of particles governed by the original stochastic differential equation. In the original stochastic differential equation, the function $f(s)$ is often referred to as the advection term, and the function $g(s)$ is often referred to as the diffusion term, and the relationship between the two determines the extent to which the system they describe is influenced by random displacements. If $g(s)$ is significantly larger than $f(s)$, the diffusion of the pdf of S overpowers its displacement but if $f(s)$ is significantly larger than $g(s)$ then the reverse is true.

In a more general form, the Fokker-Plank is motivated by the m dimensional linear system

$$\frac{d}{ds}\mathbf{s}(n) = \mathbf{b}(\mathbf{s},n) + \sigma(\mathbf{s},n)\epsilon(n)$$

where $\sigma(\mathbf{s}, n)$ is an $M \times n$ matrix and $\epsilon(s)$ is a vector of n independent white noise terms that satisfy $E[\epsilon_i(n)] = 0$ and $E[\epsilon_i(n)\epsilon_j(n+\nu)] = \delta_{ij}\delta(\nu)$, where δ_{ij} is the Kronecker delta and $\delta(n)$ is the regular delta function. The Kroenecker delta serves as an indicator function and is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The corresponding Fokker-Plank equation is

$$\frac{\partial p}{\partial n} = -\sum_{i=1}^{M} \frac{\partial}{\partial s_i}(b_i p) + \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M} \frac{\partial^2}{\partial s_i \partial s_j}(a_{ij} p)$$

where $p = p(\mathbf{s}, n)$ and (a_{ij}) is an element of the matrix A where A is defined as $A(\mathbf{s}, n) = \sigma(\mathbf{s}, n)\sigma^T(\mathbf{s}, n)$ (Zwillinger, 1989). The initial conditions for the Fokker-Plank equations are related to the initial conditions for the original linear system, so if the initial conditions for the original linear system are $\mathbf{s}(n_0) = \mathbf{z}$ then then initial conditions for the corresponding Fokker-Plank equation are $p(\mathbf{s}, n_0) = \prod_{i=1}^{M} \delta(s_i - z_i)$.

It should be noted although the Fokker-Plank equation if often hard to solve, and its solution must commonly be approximately numerically, it is particularly useful because its solution, the pdf of $\mathbf{S}$, describes the statistical characteristics of the process completely.

Although the Fokker-Plank equation was introduced specifically to study the evolution of the pdf of a non-linear differential equation it should be noted that it can be used to study the advection and diffusion of the pdf of systems in general. For example, if the stochastic differential equation that motivates the Fokker-Plank equation has no deterministic term, the corresponding Fokker-Plank equation

$$\frac{\partial p}{\partial n} = \frac{1}{2}\frac{\partial^2}{\partial s^2}(g^2(s)p)$$

describes the evolution of the pdf of the initial stochastic differential equation only in terms its diffusion. In this case the pdf would evolve in a similar way to how the pmf of the random walk in Section 2 evolved.

Conversely, if the Fokker-Plank is used to study the pdf of a normal differential equation, without a stochastic term, it will be

$$\frac{\partial p}{\partial n} = -\frac{\partial}{\partial s}(f(s)p)$$

and it will describe the pdf of the original differential equation purely in terms of its advection. In this case, the shape of the pdf would not change over time but would move along a trajectory described by the initial system, as if it were a particle released onto one of the trajectories determined by the SAVM.

6 Conclusion

In this paper the study of the advection and diffusion of uncertainty has been developed from the context of discrete time and space through to the context of continuous time and space. In Section 2, the concept of a random walk along a one dimensional lattice was discussed and the evolution of its probability mass function was explored in order to introduce idea of the evolution of uncertainty. This idea was further explored in Section 3 and in that section several examples of the evolution of the pmf for bounded random walks were given in both one and two dimensions. In Section 4, stochastic difference equations were introduced and the evolution of the pdf of a non-linear system was demonstrated using the SAVM. In Section 5 the Fokker-Plank equation was presented as a technique to study the evolution of the pdf of any type of differential equation in continuous time and continuous space.

7 References

Beichelt, Frank. *Stochastic Processes in Science, Engineering and Finance*. New York: Chapman and Hall/CRC, 2006.

Gardiner, C.W.. *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*. New York: Springer-Verlag, 1983.

Kuznetsov, L. et al. "A Method for Assimilation of Lagrangian Data." *American Meterological Society Monthly Weather Review*. 131 (2003): 2247-2260.

Lawler, Gregory F.. *Introduction to Stochastic Processes*. 2nd ed. New York: Chapman and Hall/CRC, 2006.

Shumway, Robert H. and David S. Stoffer. *Time Series Analysis and Its Applications; With R Examples*. 2nd ed. New York: Springer, 2006.

Stewart, William J.. *Introduction to the Numerical Solutions of Markov Chains*. Princeton: Princeton University Press, 1994.

Zwillinger, Daniel. *Handbook of Differential Equations*. Boston: Academic Press, Inc., 1989.

8 Appendix A

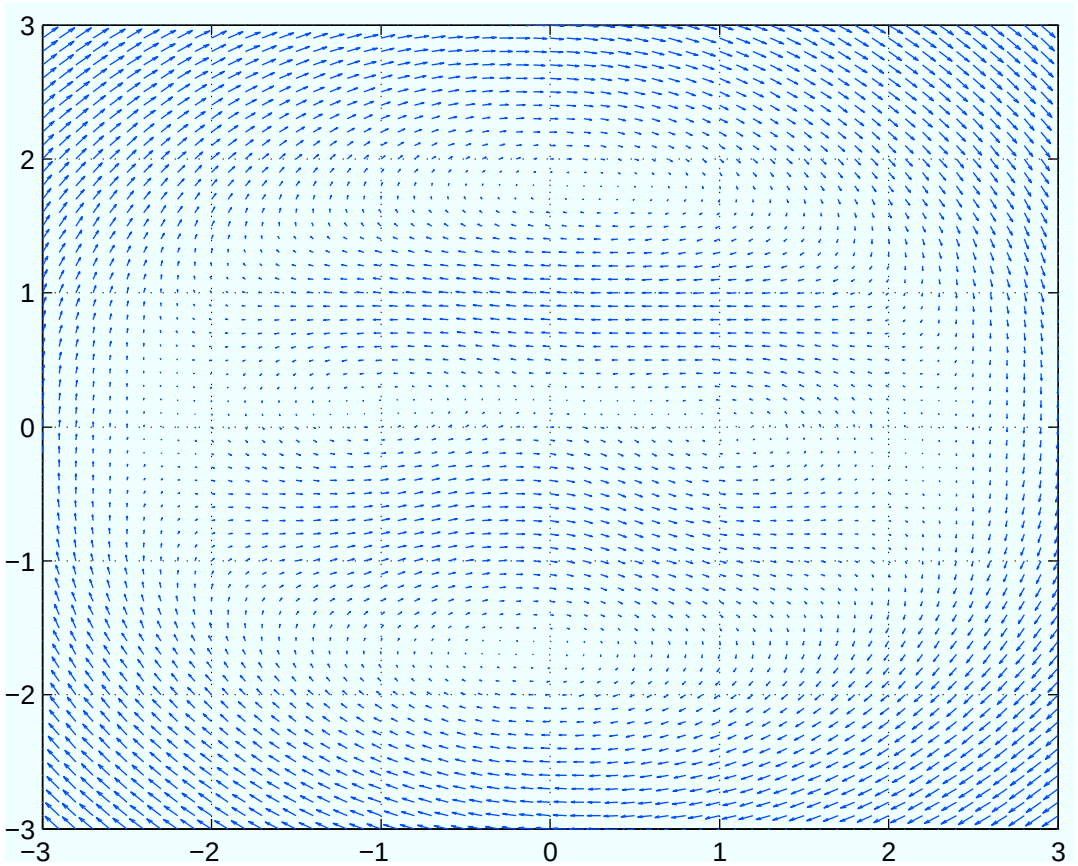


Figure 13: A vector field that represents the displacement over one time step induced by a SAVM consisting of two vortices that spin in a counterclockwise direction. Note that the centers of the two vortices are at (-1,0) and (1,0) and the appearance of two 'ghost' vortices centered at $\approx$ (-.1,-1.6) and (.1,1.6).