Midterm Solutions

1 $ty' + (t+1)y = 2te^{-t}$ $\quad y(1) = a$

a) $y' + (1+\frac{1}{t})y = 2e^{-t}$

b) $u(t) = e^{\int 1+\frac{1}{t}\,dt} = e^{t+\log t} = e^{\log t}e^{t}$

$= te^{t}$

$\boxed{u(t) = te^{t}}$ integrating factor

c) $\frac{d}{dt}(yte^{t}) = 2e^{-t}te^{t} = 2t$

$yte^{t} = t^2 + c$

$y = e^{-t}\left[t + \frac{c}{t}\right]$

$y(1) = a = e^{-1}(1+c) \Rightarrow c = ae-1$

$$\boxed{y = e^{-t}\left[t + \frac{ae-1}{t}\right]}$$

d) $\frac{1}{t}$ is singular at $t = 0$, so

critical value of a is

$$\boxed{a_0 = 1/e}$$

e) when $a < a_0$, $ae - 1 < 0$

$\Rightarrow y \to -\infty$ as $t \to 0$

when $a > a_0$, $ae - 1 > 0$

$\Rightarrow y \to \infty$ as $t \to 0$

when $a = a_0$, $ae - 1 = 0$

$\Rightarrow y \to 0$ as $t \to 0$

(2) $\frac{dy}{dx} = \frac{e^x}{2(y+1)}$ $\quad y(1) = 0$

a) $\int 2(y-1)\,dy = \int e^x\,dx$

$$\boxed{(y-1)^2 = e^x + c}$$ implicit

b) $(y-1)^2 = e^x + c$

$y(1) = 0 \Rightarrow 1 = e + c \Rightarrow c = 1 - e$

so

$(y-1)^2 = e^x + 1 - e$

$y = 1 \pm \sqrt{e^x - e + 1}$

require $y(1) = 0 \Rightarrow$ pick $-$'ve branch

$$\Rightarrow \boxed{y = 1 - \sqrt{e^x - e + 1}}$$

c) require $e^x - e + 1 > 0$

$e^x > e - 1$

$$\boxed{x > \log(e-1)}$$

d) any initial condition posed on the line $y=1$ would lead to non-uniqueness because both +'ve and -'ve branches of square root would satisfy IC

so pairs (x_0, y_0) would be

$$\boxed{(x_0, 1) \quad \text{for} \quad x_0 \in \mathbb{R}}$$

(3) $y'' + p(t)\, y' + q(t)\, y = 0 = L(y)$

suppose $L(y_1) = L(y_2) = 0$.

now

$$W = y_1 y_2' - y_2 y_1'$$

$$\frac{dW}{dt} = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1''$$

$$= y_1 y_2'' - y_2 y_1''$$

$$= y_1\left[-p(t)y_2' - q(t)y_2\right] - y_2\left[-p(t)y_1' - q(t)y_1\right]$$

$$= p(t)\left[y_1'y_2 - y_2'y_1\right]$$

$$= -p(t)\,W$$

$$\Rightarrow \quad \frac{dW}{dt} = -p(t)\,W$$

so

$$W = W_0\,\underbrace{e^{-\int p(t)\,dt}}_{\text{never } 0}$$

W_0 is constant

so $W \equiv 0$ if $W_0 = 0$, or W is never 0 if $W_0 \neq 0$ $\blacksquare$

(4) $y'' - 4y' + 5y = 0, \quad y(4) = 0, \quad y'(4) = \beta.$

C. E.: $r^2 - 4r + 5 = 0$

$$r = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

so

$$y_1 = e^{(2+i)t} \quad , \quad y_2 = e^{(2-i)t}$$

multiply y_1 by $e^{-(2+i)4}$:

$$y_1 \to e^{(2+i)(t-4)}$$

then

$$Re(y_1) = e^{2(t-4)} \cos(t-4)$$

$$Im(y_1) = e^{2(t-4)} \sin(t-4).$$

so we may write

$$y(t) = e^{2(t-4)}\left[A\cos(t-4) + B\sin(t-4)\right]$$

$$y(4) = 0 \quad \Rightarrow \quad A = 0$$

$$y'(t) = 2e^{2(t-4)}\left[A\cos(t-4) + B\sin(t-4)\right]$$

$$+ e^{2(t-4)}\left[-A\sin(t-4) + B\cos(t-4)\right]$$

$$y'(4) = 2A + B = \beta \quad \Rightarrow \quad B = \beta.$$

so

$$\boxed{y = \beta e^{2(t-4)} \sin(t-4)}$$

can also make change of variables:

let $\tau = t - 4$ and let

$$y(t) = v(t-4) = v(\tau)$$

then

$$\frac{dy}{dt} = \frac{dv}{dt} = \frac{dv}{d\tau}\underbrace{\frac{d\tau}{dt}}_{1} = \frac{dv}{d\tau}$$

similarly $\quad \dfrac{d^2y}{dt^2} = \dfrac{d^2v}{d\tau^2}$

then

$$v'' - 4v' + 5v = 0, \quad v(0) = 0, \; v'(0) = \beta$$

then solve for v

$$\rightarrow v = \beta e^{2\tau} \sin(\tau)$$

then let $\tau \rightarrow t - 4$

so

$$y = v(t-4) = \beta \sin(t-4)\, e^{2(t-4)}$$

(5) $$y'' + 9y = \begin{cases} 9(t-1), & 0 \le t \le 1 \\ \log t & t > 1 \end{cases} \qquad \begin{array}{l} y(0) = 1 \\ y'(0) = 0 \end{array}$$

on $0 \le t \le t$, $y = y_\ell$, where

$$y_\ell'' + 9y_\ell = 9(t-1), \quad y_\ell(0) = 1,$$
$$y_\ell'(0) = 0$$

homog.: $y_\ell = A\cos 3t + B\sin 3t$

particular: $y_{\ell p} = at + b$

$$y_{\ell p}' = a, \quad y_{\ell p}'' = 0$$

$$\Rightarrow 9(at + b) = 9t - 1$$

$$\Rightarrow a = 1, \quad b = -1$$

$$\Rightarrow y_{\ell p} = t - 1$$

$$\Rightarrow y_\ell = A\cos 3t + B\sin 3t + t - 1$$

$$y_\ell(0) = 1 \Rightarrow A + 0 - 1 = 1$$

$$\Rightarrow A = 2$$

$$y_\ell'(0) = 3B + 1 = 0 \Rightarrow B = -\frac{1}{3}$$

$$y_\ell(t) = 2\cos 3t - \frac{1}{3}\sin 3t + t - 1$$

$$y_\ell(1) = 2\cos 3 - \frac{1}{3}\sin 3$$

$$y_\ell'(1) = -6\sin 3 - \cos 3 + 1$$

then require continuity and differentiability in y, we must have

$$\boxed{\begin{aligned} y_r'' + 9y_r &= \log t \\ y_r(1) &= 2\cos 3 - \frac{1}{3}\sin 3 \\ y_r'(1) &= -6\sin 3 - \cos 3 + 1 \end{aligned}}$$

Use variation of paramters to solve for $y_r(t)$

b) y'' is continuous everywhere, since

$$f(t) = \begin{cases} 9(t-1) & 0 \le t \le 1 \\ \log t & t > 1 \end{cases}$$

is continuous everywhere, and

$y'' + 9y = f(t)$. By construction,

y is continuous everywhere, so

$y'' = f(t) - 9y$ must also

be continuous everywhere.

(6) $y'' - (1+i)y' + iy = t \sin t$.

C.E: $(r-i)(r-1) = 0 \Rightarrow r_1 = i,\ r_2 = 1.$

so we have

$$y_1 = e^{it}, \quad y_2 = e^{t}.$$

now

$$t \sin t = t \left[\frac{e^{it} - e^{-it}}{2i} \right]$$

$$= \underbrace{\frac{t}{2i} e^{it}}_{\text{homog. term}} - \frac{t}{2i} e^{-it}.$$

so our guess for y_p must be

$$\boxed{y_p = (a_0 + a_1 t) e^{-it} + t(b_0 + b_1 t) e^{it}}$$

b) $y'' - y = t e^{t} \cosh t$

$$= t e^{t} \left[\frac{e^{t} + e^{-t}}{2} \right]$$

$$= \frac{t e^{2t}}{2} + \frac{t}{2}.$$

homog: C.E: $r^2 - 1 = 0, \Rightarrow r = \pm 1$

so

$$y_1 = e^{t}, \quad y_2 = e^{-t}.$$

then our guess must be

$$\boxed{y_p = a_0 + a t + (b_0 + b_1 t)e^{2t}}$$