Sysems of ODES (5.1, 5.2, 5.3).

notation

- t will be indep. variable "time"
- x, y, u, v will be dep. variables (~~small~~ scalar)
- $\vec{x}, \vec{y}, \vec{v}$ etc will be vectors
- $\underline{\underline{A}}$ matrix
- eve eigenvalue
- evr eigenvector

sysems of ODE's have the form

$$\left.\begin{aligned} x' &= f(x, y, t) \\ y' &= g(x, y, t). \end{aligned}\right\} \text{ if } f, g \text{ indep of } t, \text{ the system is autonomous.}$$

ex

$$x' = 2x + 4y + \sin t$$

$$y' = -5x + 7y + 2$$

applications

- write

$$mx'' + cx' + kx = f(t), \quad x(0) = \alpha$$
$$x'(0) = \beta$$

as a system of 2 1st order ODE's.

let $y = x' \Rightarrow y' = x'' = \frac{f(t)}{m} - \frac{cx'}{m} - \frac{kx}{m}$

$$= \frac{f(t)}{m} - \frac{c}{m} y - \frac{k}{m} x.$$

$$\frac{dx}{dt} = 0 \cdot x + 1 \cdot y \qquad x(0) = \alpha$$

$$\frac{dy}{dt} = \frac{-k}{m} x - \frac{c}{m} y + \frac{f(t)}{m}. \qquad y(0) = \beta.$$

then

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}}_{=M} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t)/m \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{v}.$$

$$\frac{d\vec{v}}{dt} = \underline{\underline{M}}\,\vec{v} + \vec{g}(t).$$

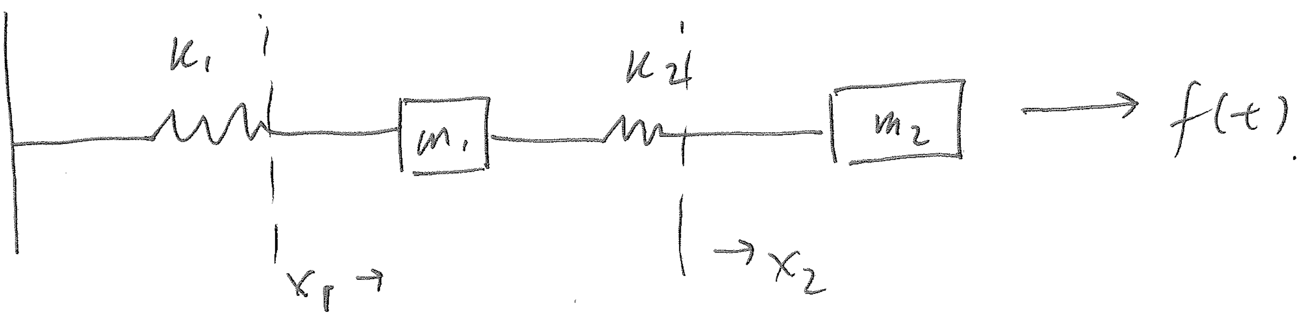
same for higher ODEs:

$$x''' + 3x'' + 4x' + 2x = 0.$$

$$x' = y$$
$$y' = w = x''$$
$$w' = x''' = -3x'' - 4x' - 2x$$
$$= -3w - 4y - 2x.$$

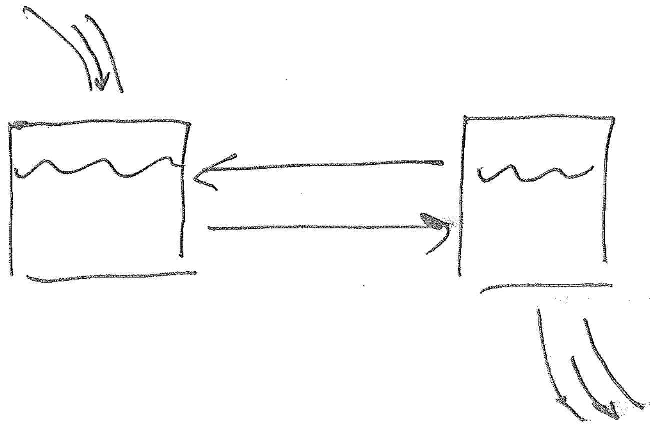
$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ w \end{pmatrix}$$ linear.

- coupled spring mass systems



$$m_1 x_1'' = -k_1 x + k_2(x_2 - x_1)$$ linear
$$m_2 x_2'' = -k_2(x_2 - x_1) + f(t).$$

- water tanks with exchange



(see p. 327) (linear)

- predator prey models (e.g. Lotka-Volterra)

let

$x(t)$ = # of prey at time t.

$y(t)$ = # of predators at time t.

then we have

(nonlinear)

$$\frac{dx}{dt} = \underbrace{\alpha x}_{\text{birth}} - \underbrace{\beta xy}_{\text{death due to interaction with predator}}.$$

$$\frac{dy}{dt} = \underbrace{-\gamma y}_{\text{death}} + \underbrace{\delta xy}_{\text{birth (more prey} \Rightarrow \text{more reproduction)}}$$

first look at easy (but general example)

$$m x'' + c\dot{x} + kx = 0 \quad x(0) = \alpha, \; x'(0) = \beta \quad (1)$$

$$x = e^{rt}$$

$$\hookrightarrow \; m r^2 + c r + k = 0 \qquad (\text{C.E.})$$

$$r_{\pm} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

then

$$x(t) = A e^{r_+ t} + B e^{r_- t}.$$

$$x(0) = A + B = \alpha.$$

$$x'(0) = r_+ A + r_- B = \beta.$$

$$A = \frac{\begin{vmatrix} \alpha & 1 \\ \beta & r_- \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_+ & r_- \end{vmatrix}}, \qquad B = \frac{\begin{vmatrix} 1 & \alpha \\ r_+ & \beta \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_+ & r_- \end{vmatrix}}.$$

then we have

$$x(t) = A e^{r_+ t} + B e^{r_- t}$$

$$x'(t) = r_+ A e^{r_+ t} + r_- B e^{r_- t}.$$

or

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = A \begin{pmatrix} 1 \\ r_+ \end{pmatrix} e^{r_+ t} + B \begin{pmatrix} 1 \\ r_- \end{pmatrix} e^{r_- t}.$$

this suggest that we can solve (1) as a system of 2 1^{st} order ODE's. 1 eqn for x, 1 eqn for $x' \equiv y$.

now solve as a system:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \equiv \vec{v}, \quad \underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix}$$

$$\frac{d\vec{v}}{dt} = \underline{A}\,\vec{v} \;; \quad \vec{v}(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

looks

$$\cancel{\frac{dy}{dt}} \quad \frac{du}{dt} = ru, \qquad u(0) = C.$$

solution was $u = Ce^{rt}$.

this suggests we make the guess

$$\vec{v} = \vec{\xi} e^{\lambda t} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t}.$$

where $\vec{\xi}$ and λ are to be determined.

now

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} \xi_1 \lambda e^{\lambda t} \\ \xi_2 \lambda e^{\lambda t} \end{pmatrix} = \lambda \vec{\xi} e^{\lambda t}.$$

now we get

$$\lambda \vec{\xi} \cancel{e^{\lambda t}} = \underline{A}\, \vec{\xi} \cancel{e^{\lambda t}}.$$

so

$$A\vec{\xi} = \lambda \vec{\xi}$$

$\not{\!10}$

so λ is an eve of $\underline{\underline{A}}$, $\vec{\xi}$ is the corresponding evr.

so from

$$\underline{\underline{A}}\,\vec{\xi} = \lambda \vec{\xi}$$

we rearrange to obtain

$$\underline{\underline{A}}\,\vec{\xi} - \lambda \vec{\xi} = 0$$

$$\underline{\underline{A}}\,\vec{\xi} - \lambda \underline{\underline{I}}\,\vec{\xi} = 0 \qquad \underline{\underline{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\vec{\xi} = 0.$$

since we require $\vec{\xi} \neq \vec{0}$, $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ must have a nontrivial nullspace $\Rightarrow$ yields condition for λ ($\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\xi = 0.$)

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \left[\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right].$$

$$= \det \begin{pmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{pmatrix} = 0.$$

$$(-\lambda)\left(-\frac{c}{m} - \lambda\right) + \frac{k}{m} = 0.$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$m\lambda^2 + c\lambda + k = 0. \qquad (\text{C.E.})$$

$$\lambda_\pm = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

now to find evr's, $\vec{\xi}^+$ and $\vec{\xi}^-$ corresponding to λ^+ and λ^-, use

$$(A - \lambda_\pm I)\vec{\xi}^\pm = 0 \quad \Big| \text{ let } \vec{\xi}^\pm = \begin{pmatrix} \xi_1^\pm \\ \xi_2^\pm \end{pmatrix}.$$

$$\begin{pmatrix} -\lambda_\pm & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda_\pm \end{pmatrix} \begin{pmatrix} \xi_1^\pm \\ \xi_2^\pm \end{pmatrix} = 0.$$

rows are l.d., so only use 1 row.

(A10)

$$-\lambda_\pm \xi_1^\pm + \xi_2^\pm = 0.$$

take $\xi_1^\pm = 1$ wlog. (evr are determined only up to a multiplicative constant)

then $\xi_2^\pm = \lambda_\pm$

therefore, we have two eigenpairs

$$\lambda_+, \vec{\xi}^+, \quad \lambda_-, \vec{\xi}^- \qquad \Big| \ \vec{v} = \vec{\xi} e^{\lambda t}.$$

$\vec{v}$ is then a linear combination of the two:

$$\vec{v} = c_+ \vec{\xi}^+ e^{\lambda_+ t} + c_- \vec{\xi}^- e^{\lambda_- t}$$

$$= c_+ \begin{pmatrix} 1 \\ \lambda^+ \end{pmatrix} e^{\lambda_+ t} + c_- \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix} e^{\lambda_- t}.$$

to find c_+ and c_-, apply IC's:

$$\vec{v}(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = c^+ \begin{pmatrix} 1 \\ \lambda^+ \end{pmatrix} + c^- \begin{pmatrix} 1 \\ \lambda^- \end{pmatrix}.$$

$$\text{or } \begin{pmatrix} 1 & 1 \\ \lambda^+ & \lambda^- \end{pmatrix} \begin{pmatrix} c^+ \\ c^- \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$c^{+} = \frac{\begin{vmatrix} \alpha & 1 \\ \beta & \lambda^{-} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \lambda^{+} & \lambda^{-} \end{vmatrix}}, \quad c^{-} = \frac{\begin{vmatrix} 1 & \alpha \\ \lambda^{+} & \beta \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ \lambda^{+} & \lambda^{-} \end{vmatrix}}$$

compare to solution of 2^{nd} order equation
(the same, of course $\lambda^{\pm} \to r^{\pm}$)

key steps for solving $\frac{d\vec{v}}{dt} = \underline{\underline{A}}\,\vec{v}$:

1) make guess $\vec{v} = \vec{\xi} e^{\lambda t}$.

2) find e've's by by solving $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$
↳ obtain C.E., in general, more than one

3) for each λ, find $\vec{\xi}$
(solve undetermined system → $\vec{\xi}$ in non-unique)

4) write
$$\vec{v} = c_1 \vec{\xi}_1 e^{\lambda_1 t} + c_2 \vec{\xi}_2 e^{\lambda_2 t} + \dots$$

(if two distinct evrs exist).

5) find c_1, c_2 by solving $\vec{v}(0) = (\vec{\xi}^+ \ \vec{\xi}^-)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (\vec{\xi}^+ \ \vec{\xi}^-)^{-1} \vec{v}(0).$$