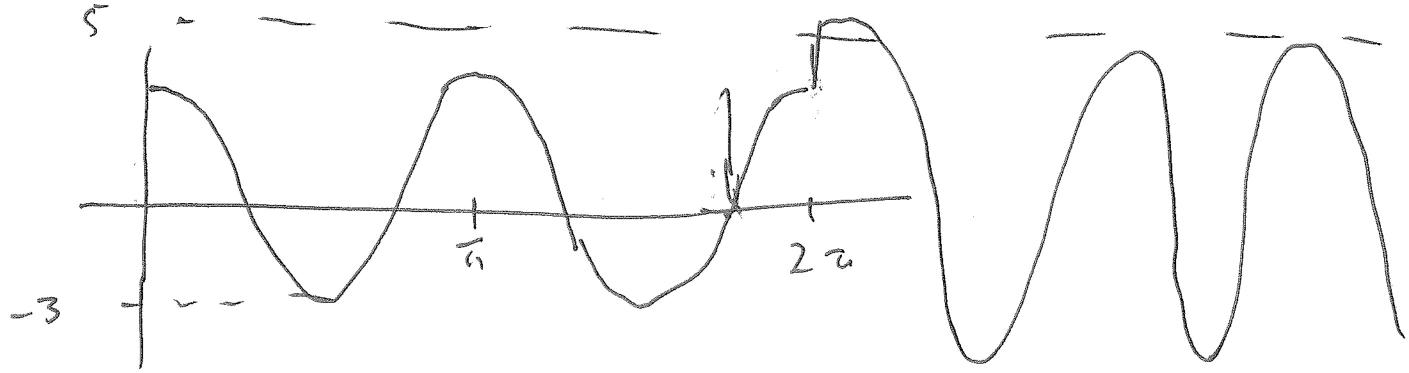
Last time we solved

$$x'' + 4x = 8\,\delta(t - 2\pi) \qquad x(0) = 3,\ x'(0) = 0$$

$$x(t) = \begin{cases} 3\cos 2t & 0 < t < 2\pi \\ \underbrace{3\cos 2t + 4\sin 2t}_{} & t > 2\pi. \end{cases}$$

$\hookrightarrow 5\cos(2t - 0.9273)$



question: how would you time the kick to make all motion cease for all subsequent time?

<u>Ex</u> (p.321).

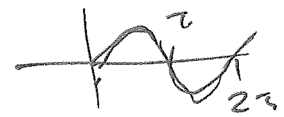
$$x'' + x = \sum_{n=0}^{\infty} \delta(t - n\pi), \quad x(0) = x'(0) = 0.$$

apply $\mathcal{L}(\cdot)$:

$$(s^2+1)\,X(s) = \sum_{n=0}^{\infty} e^{-n\pi s}$$

$$X(s) = \frac{\sum_{n=0}^{\infty} e^{-n\pi s}}{s^2+1} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \cdots$$

$$x(t) = \sum_{n=0}^{\infty} u(t-n\pi)\sin(t-n\pi)$$



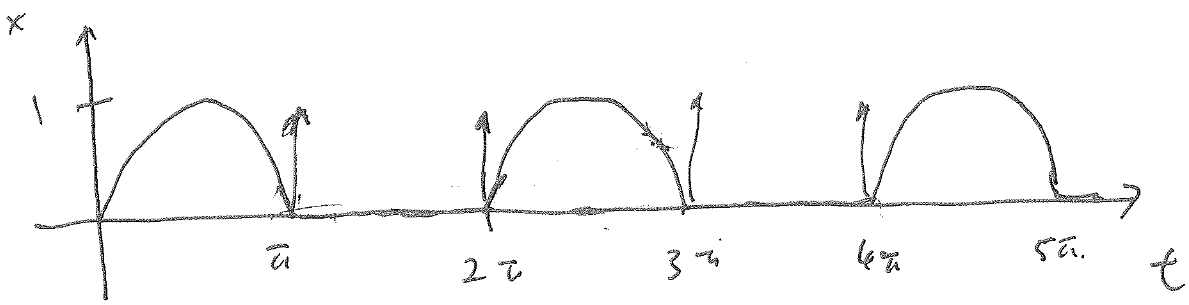
$$\sin(t-n\pi) = (-1)^n \sin t.$$

$$x(t) = \sum_{n=0}^{\infty} u(t-n\pi)(-1)^n \sin t.$$

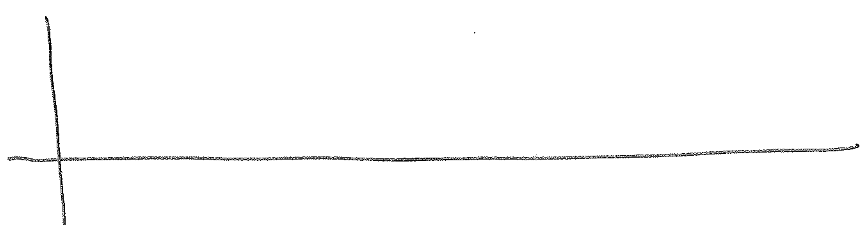
- $0<t<\pi$: $x = \sin t.$
- $\cancel{\pi < t < 2\pi : x = \sin t}$

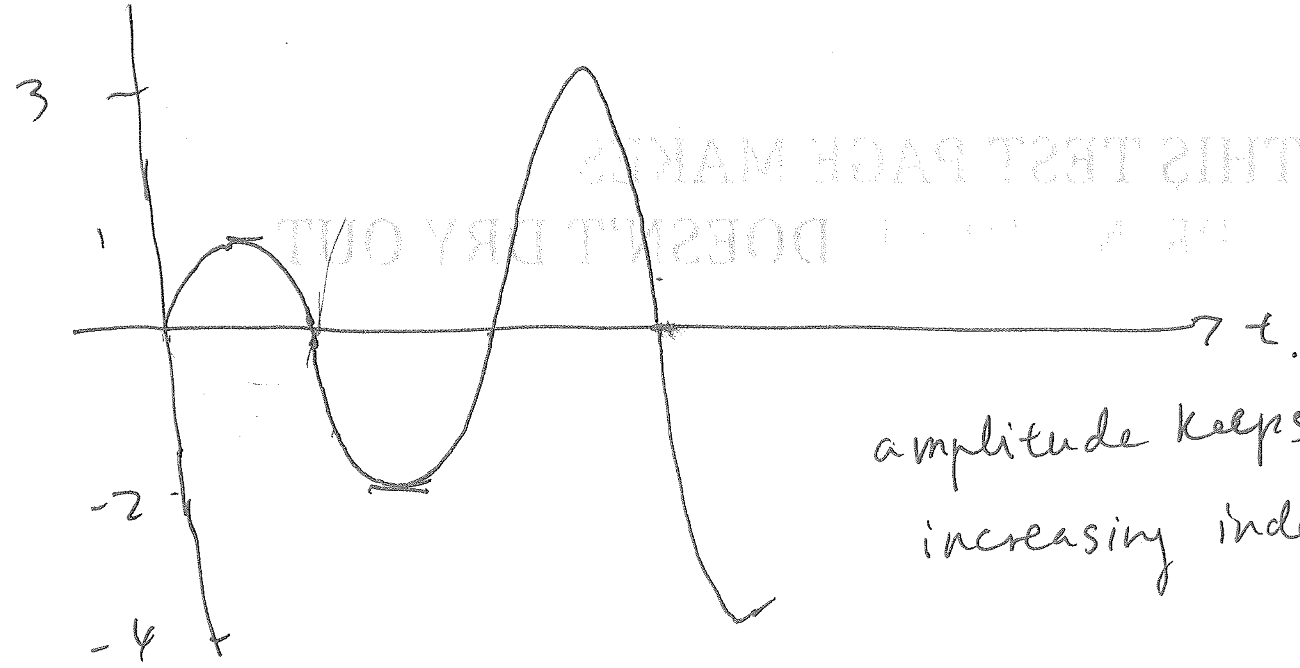
$\pi < t < 2\pi$: $x = \sin t - \sin t = 0.$

$2\pi < t < 3\pi$: $x = \sin t - \sin t + \sin t = \sin t.$



what if $f(t) = \sum_{n=0}^{\infty} (-1)^n \delta(t-n\pi)$.





amplitude keeps increasing indefinitely

<u>Duhamel's principle</u>

recall for $mx'' + cx' + kx = f(t)$ with 0 IC's,

then

$$X(s) = \frac{F(s)}{Z(s)}, \quad Z(s) = ms^2 + cs + k$$

then by convolution,

$$x(t) = \int_0^t f(\tau)\, \zeta(t-\tau)\, d\tau,$$

where $\zeta(t) = \mathcal{L}^{-1}\left(\frac{1}{Z(s)} \right)$

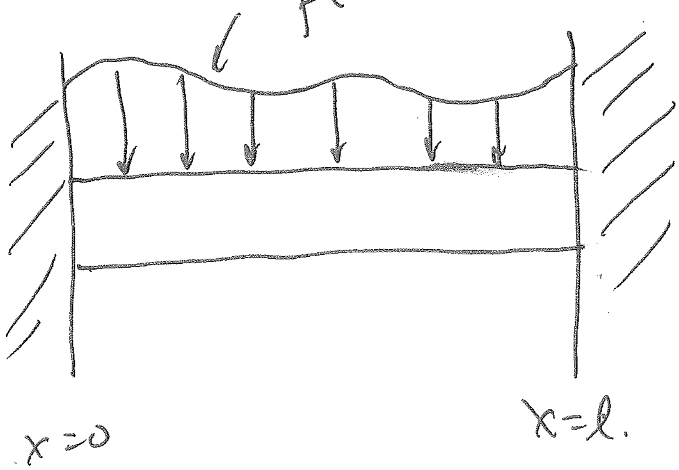
what is $\zeta(t)$? it is the unit impulse response; that is, the solution to

$$m\zeta'' + c\zeta' + k\zeta = \delta(t), \quad \zeta(0) = \zeta'(0) = 0$$

analogy to Green's function for boundary value problem

$f(x)$ ← weight distribution

EX



the deflection of the beam $u(x)$ is given by

$$-\sigma u''(x) = f(x)$$
$$u(0) = u(l) = 0$$

σ is a measure of the stiffness of the beam.
(constant)

multiply by G and integrate from $x=0 \to L$.

$$-\int_0^l G\,\sigma u''\,dx = \int_0^l G f(x)\,dx$$

integrate by parts:

$$-\sigma\left[G u'\Big|_0^l - \int_0^l G' u' \, dx\right] = \int_0^l G f(x)\, dx.$$

$$-\sigma\left[G u'\Big|_0^l - \cancel{G' u\Big|_0^l} + \int_0^l u G'' dx\right] = \int_0^l G f(x)\, dx.$$

choose $G(l) = G(0) = 0$

then we have.

$$\int_0^l u(-\sigma G'')dx = \int_0^l G f(x)\, dx.$$

let

$$-\sigma G'' = \delta(x-\xi); \quad G_\xi = G(x;\xi).$$

$G(0) = G(l) = 0$

$$u(\xi) = \int_0^l G(x|\xi) f(x)\, dx.$$

G is called the Green's fcn for the d^2/dx^2 (Laplacian).

relationship between δ-fcn and step fcn

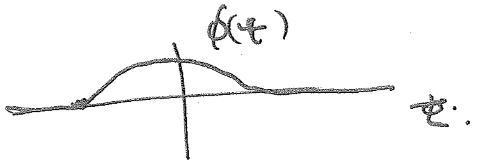
recall

$$\int_{-\infty}^{t} \delta(t-a)\,dt = u(t-a).$$

So formally,

$$\delta(t-a) = \frac{d}{dt} u(t-a).$$

in the "weak sense" because $u(t-a)$ has no strong derivative at $t=a$. Let $\phi(t)$ be an infinitely differentiable "test fcn" indentically zero outside some bounded interval



Then

$$\int_{-\infty}^{\infty} \boxed{\frac{d}{dt} u(t-a)}\,\phi(t)\,dt = \cancel{\phi(t)\,u(t-a)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} u(t-a)\,\phi'(t)\,dt.$$

$$= -\int_a^\infty 1 \cdot \phi'(t)\,dt = -[\phi(\infty) - \phi(a)]$$

$$= \phi(a) = \int_{-\infty}^{\infty} \boxed{\delta(t-a)}\,\phi(t)\,dt.$$

so we can say $\frac{d}{dt} u(t-a) = \delta(t-a)$

in ~~the~~ weak sense.

see book for a different derivation.

Initial value theorem

$$\lim_{s\to\infty} sY(s) = y(0)$$

large s $\leftrightarrow$ small t in ~~physical~~ temporal space.

Final value theorem

$$\lim_{s\to 0} sY(s) = \lim_{t\to\infty} y(t).$$