rule 2

if $Q(s) = \left((s-a)^2 + b^2\right)^n q(s)$, where $q(a \pm ib) \neq 0$, then the portion of the p.f.e corresponding to $(s-a)^2 + b^2$, would be

$$\frac{A_1 s + B_1}{(s-a)^2 + b^2} + \frac{A_2 s + B_2}{\left[(s-a)^2 + b^2\right]^2} + \dots + \frac{A_n s + B_n}{\left[(s-a)^2 + b^2\right]^n}$$

rule 2b

if $Q(s) = (s^2 + bs + c)^n q(s)$, complete the square to obtain

$$Q(s) = \left[\left(s + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}\right]^n q(s)$$

$$c - \frac{b^2}{4} > 0 \quad \text{or else use rule 1}$$

and apply rule 2C + translation rule to invert.

rule 2C

to invert

$$\frac{As+B}{(s-a)^2+b^2},$$

write as

$$\frac{A(s-a)+B+aA}{(s-a)^2+b^2}$$

$$= \frac{A(s-a)}{(s-a)^2+b^2} + \frac{B+aA}{(s-a)^2+b^2}$$

exponentially decaying/growing cosine

...
...
sine

apply translation rule

Ex

Solve $y'' + 4y' + 8y = 0$, $y(0) = 1$, $y'(0) = 0$

apply $\mathcal{L}(\cdot)$:

$$\underbrace{s^2 Y(s) - s y(0) - \cancel{y'(0)}}_{\mathcal{L}(y'')} + 4[\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}]$$

$$+ 8Y(s) = 0$$

$$(s^2 + 4s + 8)Y(s) = s + 4$$

$$Y(s) = \frac{s+4}{s^2+4s+8}$$

$$= \frac{s+4}{\underbrace{s^2+4s+4}+4} = \frac{s+4}{(s+2)^2+4}$$

$$= \frac{s+2+2}{(s+2)^2+2^2} = \frac{s+2}{(s+2)^2+2^2} + \frac{2}{(s+2)^2+2^2}$$

use:

$$\mathcal{L}(\cos mt) = \frac{s}{s^2+m^2}$$

$$\mathcal{L}(\sin mt) = \frac{m}{s^2+m^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+2^2}\right)$$

$$+ \mathcal{L}^{-1}\left(\frac{2}{(s+2)^2+2^2}\right).$$

$$= e^{-2t}\cos 2t + e^{-2t}\sin 2t.$$

here, used $\mathcal{L}(e^{at}f(t)) = F(s-a)$

Ex $y'' - 2y' + 9y = 0$, $y(0) = 0$, $y'(0) = 1$

$$y = \frac{1}{\sqrt{8}} e^{t} \sin\sqrt{8}\,t.$$

on your own

<u>EY</u> (rule 1 example)

solve

$$y'' - y' - 2y = 4t^2, \; y(0) = 0, \; y'(0) = 0$$

(tedious...)

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

apply $\mathcal{L}(\cdot)$:

$$(s^2 - s - 2)Y(s) = \frac{8}{s^3}$$

$$Y(s) = \frac{8}{s^3(s-2)(s+1)}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} + \frac{E}{s-2} = \frac{8}{s^3(s-2)(s+1)}$$

D: multiply by $s+1$, then set $s = -1$

$$D = \frac{8}{3}$$

E: multiply by $s-2$, then set $s = 2$

$$E = \frac{8}{2^3 \cdot 3} = \frac{1}{3}$$

C: $\times$ by s^3, set $s = 0$

$$C = -4$$

B: multiply by s^3:

$$As^2 + Bs + C + s^3 [\ldots] = \frac{8}{(s-2)(s+1)}$$

differentiate w.r.t. s:

$$2As + B + s^2 [\ldots] = -\frac{8(2s-1)}{(s-2)^2(s+1)^2}$$

set $s = 0$:

$$B = 2$$

A: differentiate again, set $s = 0$ ...

$$A = -3$$

$$Y(s) = \frac{-3}{s} + \frac{2}{s2} - \frac{4}{s^3} + \frac{8}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$

use : $\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$,

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$:

$$y(t) = \underbrace{-3 + 2t - 2t^2}_{\text{particular soln}} + \underbrace{\frac{8}{3} e^{-t} + \frac{1}{3} e^{2t}}_{\text{homogeneous part}}.$$

products, integrals, and derivatives of transforms (4.4)

Theorem 1 - convolution

if $F(s) = \mathcal{L}(f)$, $G(s) = \mathcal{L}(g)$ both exist, then

$$\mathcal{L}^{-1}\left(F(s)\,G(s)\right) = (f * g)(t)$$

$$\equiv \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

convolution integral

time-dependent weighted average of
f with ~~respe~~ weighting g.

<u>proof</u>:

$$F(s) = \int_0^\infty e^{-su} f(u)\,du$$

$$G(s) = \int_0^\infty e^{-sv} f(v)\,dv$$

want to show

$$F(s)\,G(s) = \mathcal{L}\left\{ \int_0^t f(\tau)\,g(t-\tau)\,d\tau \right\}$$

$$= \int_0^\infty e^{-st} \left[\int_0^t f(\tau)\,g(t-\tau)\,d\tau \right] dt$$

Laplace transform of convolution

$$F(s)\,G(s) = \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv$$

$$= \int_0^\infty f(u)\,e^{-su} \left[\int_0^\infty e^{-sv} g(v)\,dv \right] du$$

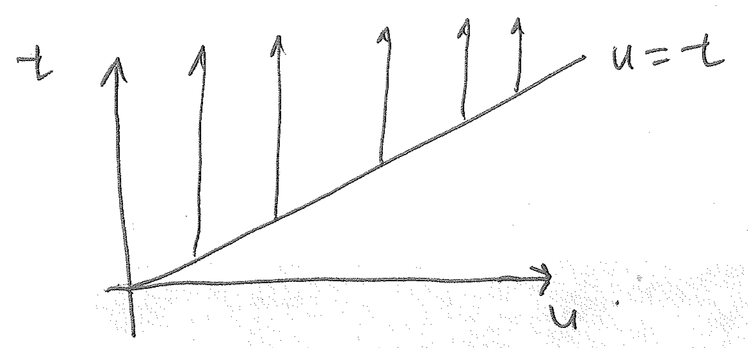
$$= \int_0^\infty f(u) \left[\int_0^\infty e^{-s(u+v)} g(v)\,dv \right] du$$

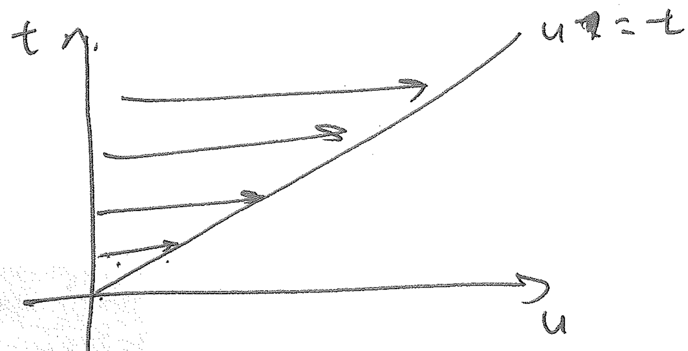
change variables: let $u+v=t$

$\Rightarrow v = t-u$, $dv = dt$.

$t: u \to \infty$, as $v: 0 \to \infty$.

$$F(s)G(s) = \int_{u=0}^{\infty} f(u) \int_{t=u}^{\infty} e^{-st} g(t-u)\,dt\,du$$





$$F(s)G(s) = \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} f(u) g(t-u)\,du\,dt$$

$$= \int_{t=0}^{\infty} e^{-st} \left[\int_{u=0}^{t} f(u) g(t-u)\,du \right] dt.$$

$\Rightarrow F(s)G(s) = \mathcal{L}(f*g)$

or

$$f * g(t) = \mathcal{L}^{-1}(F(s)\,G(s))$$

use to invert transform that is a product of two functions

<u>properties of convolution</u>

① $f * g = g * f$ (commutative)

② $f * (g_1 + g_2) = f * g_1 + f * g_2$ (distributive)

③ $(f * g) * h = f * (g * h)$ (associative)

④ $f * 0 = 0$

① proof:

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

let $z = t - \tau$, $\tau = t - z$ $d\tau = -dz$

then

$$(f*g)(t) = -\int_{z=t}^{z=0} f(t-z)\, g(z)\, dz$$

$$= \int_0^t g(z)\, f(t-z)\, dz$$

$$= (g*f)(t)$$

v v

<u>Ex</u> find inverse of

$$H(s) = \frac{a}{s^2(s^2+a^2)} = \frac{1}{s^2} \cdot \frac{a}{s^2+a^2}$$

$$\uparrow F \qquad \uparrow G.$$

$$f(t) = \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t.$$

$$g(t) = \mathcal{L}^{-1}(G) = \mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at.$$

by Thm 1,

$$h(t) = (g*f)(t) = \int_0^t \sin a\tau\, (t-\tau)\, d\tau$$

$$= t\int_0^t \sin a\tau\, d\tau - \int_0^t \tau \sin a\tau\, d\tau$$

$$= -\frac{t}{a}\cos a\tau\Big|_0^t - \left[\cancel{-\frac{\tau}{a}\cos a\tau\Big|_0^t} + \frac{1}{a}\int_0^t \cos a\tau\,d\tau\right]$$

$$= -\frac{t}{a}\cancel{\cos at} + \frac{t}{a} - \left[\cancel{-\frac{t}{a}\cos at} + \frac{1}{a^2}\sin at\right]$$

$$= \frac{t}{a} - \frac{1}{a^2}\sin at.$$

can also do by partial fractions

<u>Ex</u> $y'' + 4y = g(t), \quad y(0) = 3, \; y'(0) = -1$

$\mathcal{L}(\cdot)$:

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = G(s)$$

$$(s^2+4)Y(s) = 3s - 1 + G(s)$$

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}$$

$$= 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2G(s)}{s^2+4}$$

$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin(2(t-\tau)) \cdot g(\tau)\, d\tau$$

know the solution for <u>any</u> $g(t)$

recall, for $mx'' + cx' + kx = f(t)$,

$$X(s) = \frac{I(s)}{Z_1(s)} + \frac{F(s)}{Z_1(s)}$$

information on non-homogeneity

where

$$Z_1(s) = ms^2 + cs + k$$

$$I(s) = \ldots \quad \text{initial conditions}$$

then particular solution is

$$x_p(t) = \int_0^t f(\tau)\, \zeta(t-\tau)\, d\tau$$

where

$$\zeta(t) = \mathcal{L}^{-1}\left(\frac{1}{Z_1(s)}\right)$$

Theorem 2 - derivative of trans form

if $f(t)$ is piece-wise cont's for $t \geq 0$, and exponential order, then

$$\mathcal{L}(-t f(t)) = F'(s) \quad \text{where}$$

$$\mathcal{L}(f) = F(s)$$

[multiplication by $-t \Rightarrow$ derivative w.r.t s]

use to compute $\mathcal{L}(-t f(t))$ if $\mathcal{L}(f)$ is known, ie, compute $\mathcal{L}(f)$ then diff. w.r.t. s

or

$$f(t) = \frac{-1}{t} \mathcal{L}^{-1}(F'(s))$$

use this to invert $F(s)$ if $\mathcal{L}^{-1}(F'(s))$ is known. ie, compute $\mathcal{L}^{-1}(F'(s))$ and multiply by $\frac{-1}{t}$

proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt.$$

$$= \int_0^\infty e^{-st}\left[-t f(t)\right] dt$$

$$= \mathcal{L}(-t f(t)) \qquad \square$$

in general,

$$\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$$

where $F(s) = \mathcal{L}(f)$

aside: note analogy to moment generating function

$$M(s) = \int_{-\infty}^{\infty} e^{st} f(t)\, dt.$$

[n^{th} moment of $f(t)$ is $\int_{-\infty}^{\infty} t^n f(t)\, dt = m_n$]

$$\frac{d^n}{ds^n} M(s) = \int_{-\infty}^{\infty} e^{+st}\, t^n f(t)\, dt$$

$$M^{(n)}(0) = \int_{-\infty}^{\infty} t^n f(t)\, dt = m_n.$$

why is this true.

because

$$M(s) = \int_{-\infty}^{\infty} \left[1 + st + \frac{1}{2!}(st)^2 + \frac{1}{3!}(st)^3 + \dots\right] f(t)\,dt$$

$$= m_0 + s m_1 + \frac{s^2}{2} m_2 + \dots\dots$$

<u>Ex</u>

find $\mathcal{L}(t^2 e^{at})$

use $\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$

here, $n=2$, $f(t) = e^{at}$, $F(s) = \frac{1}{s-a}$

$$\Rightarrow \mathcal{L}(t^2 e^{at}) = (-1)^2 \frac{d^2}{ds^2} \frac{1}{s-a}$$

$$= \frac{2}{(s-a)^3}$$

or use translation thm:

$$\mathcal{L}(t^2) = \frac{2}{s^3}$$

$$\mathcal{L}(e^{at} t^2) = \frac{2}{(s-a)^3}$$

<u>EX</u> <u>p.299</u>

find $\mathcal{L}^{-1}(\tan^{-1}(1/s)) = f(t)$.

$F(s)$ is difficult to invert.

but can use

$$f(t) = -\frac{1}{t} \mathcal{L}^{-1}(F'(s))$$

$$F(s) = \tan^{-1}(1/s)$$

$$F'(s) = \frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right) = \frac{-1}{1+s^2}$$

$$\Rightarrow f(t) = \frac{-1}{t} \mathcal{L}^{-1}\left(\frac{-1}{1+s^2}\right)$$

$$= \frac{\sin t}{t}$$

<u>Thm 3</u> - integral of transform

suppose $f(t)$ is piecewise cont's and of exp. order, and $\lim_{t \to 0^+} \frac{f(t)}{t}$ exists and is finite, then

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\, d\sigma$$

or

$$f(t) = t\, \mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\, d\sigma\right\}.$$

proof:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

$$\int_s^\infty F(\sigma)\,d\sigma = \int_{\sigma=s}^\infty \int_{t=0}^\infty e^{-\sigma t} f(t)\,dt\,d\sigma$$

$$= \int_{t=0}^\infty \int_{\sigma=s}^\infty e^{-\sigma t} f(t)\,d\sigma\,dt.$$

$$= \int_{t=0}^\infty -\frac{1}{t} e^{-\sigma t}\Big|_{\sigma=s}^\infty f(t)\,dt$$

$$= \int_{t=0}^\infty \frac{1}{t} e^{-st} f(t)\,dt = \mathcal{L}\left(\frac{f(t)}{t}\right).$$

$\square$

EX compute $\mathcal{L}\left(\frac{\sin t}{t}\right)$

first: $\lim_{t \to 0^+} \frac{\sin t}{t} = 1$. so condition of thm 3 is satisfied.

$$\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \mathcal{L}(\sin t)\, d\sigma.$$

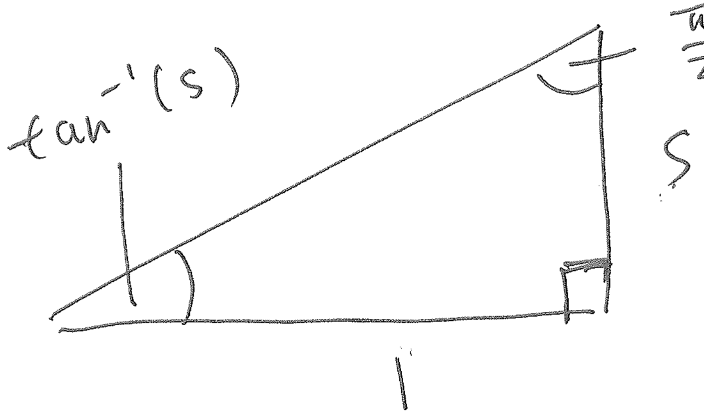
$$\mathcal{L}(\sin t) = \frac{1}{\sigma^2+1}$$

$$\Rightarrow \mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{\sigma^2+1}\, d\sigma$$

$$= \tan^{-1}\sigma \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}(s)$$

$$= \tan^{-1}(1/s) \quad \text{for} \quad s>0$$

$\uparrow$ $s>0$ for convergence.



$$\frac{\pi}{2} - \tan^{-1}(s) = \tan^{-1}(1/s)$$

<u>Ex</u> compute $\mathcal{L}^{-1}\left(\boxed{\frac{2s}{(s^2-1)^2}}\right) = f(t)$

$\underbrace{\qquad}_{\uparrow F(s)}$.

p.301

(P54)

Use

$$f(t) = t\,\mathcal{L}^{-1}\left\{ \int_s^\infty F(\sigma)\,d\sigma \right\}.$$

$$\rightarrow \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{2\sigma}{(\sigma^2-1)^2}\,d\sigma.$$

$$= \frac{-1}{\sigma^2-1}\Big|_s^\infty = \frac{1}{s^2-1}$$

$$f(t) = t\,\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = t\sinh t.$$