Math/Stat 2300 Modeling with Differential Equations

Some Terminology

The problem of finding a solution to a differential equation

$$\frac{dy}{dx} = f(y, x)$$

that satisfies

$$y(x_0) = y_0$$

is called an **initial value problem**.

A differential equation where $\frac{dy}{dx}$ is a function of y only:

$$\frac{dy}{dx} = f(y)$$

is called **autonomous**.

If $\frac{dy}{dx} = f(y)$ is an autonomous differential equation, then the values of y for which

$$\frac{dy}{dx} = 0$$

are called **equilibrium values**.

Consider

$$\frac{dy}{dx} = f(x, y).$$

Each time we specify an initial condition $y(x_0) = y_0$ for the differential equation, the **solution curve** (the graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there.

We can picture these slopes by drawing a short line segment at various points (x, y) with slope $f(x, y)$. Such graphs are called **vector fields** or **direction field**.

Each segment has the same slope as the solution curve through (x, y) and thus is tangent to the curve there.

Autonomous Differential Equations: Phase line diagrams

A **phase line diagram** is a number line with the equilibrium values, with arrows indicating the sign of y'.

Example Consider

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

Equilibrium values:

$$\frac{dy}{dx} = 0 \Longrightarrow y = -1, \;\; y = 2$$

Phase line diagram

$$
\begin{array}{ccccc}
y' > 0 & & y' < 0 & & y' > 0 \\
\longrightarrow & & \longleftarrow & & \longrightarrow \\
\hline
 & y = -1 & & y = 2 &
\end{array}
$$

On the interval $y < -1$, we try $y = -2$:

$$\frac{dy}{dx} = (-2+1)(-2-2) = 4 > 0$$

On the interval $-1 < y < 2$, we try $y = 0$:

$$\frac{dy}{dx} = (0+1)(0-2) = -2 < 0$$

On the interval $y > 2$, we try $y = 3$:

$$\frac{dy}{dx} = (3+1)(3-2) = 4 > 0$$

What might the phase line diagram indicate about the stability of these equilibrium?
arrows toward equilibrium $\implies$ stable
arrows away from equilibrium $\implies$ unstable

Constructing Graphical Solutions of Autonomous Differential Equations

- draw a phase line diagram
- identify and label where $y' > 0$ and $y' < 0$
- identify and label where $y' > 0$ and $y' < 0$
- sketch solution curves

Example. Consider

$$\frac{dy}{dx} = (y+1)(y-2)$$

We already did the phase line diagram (see above). Now, we look at the second derivative:

$$y'' = 2yy' - y' = (2y-1)y' = (2y-1)(y+1)(y-2)$$

Where is $y'' = 0$? $y = -1$, $y = 2$, $y = \frac{1}{2}$

$$
\begin{array}{ccccccc}
y'' < 0 & & y'' > 0 & & y'' < 0 & & y'' > 0 \\
y' > 0 & & y' < 0 & & y' < 0 & & y' > 0 \\
\hline
 & y = -1 & & y = \frac{1}{2} & & y = 2 &
\end{array}
$$

Newton's Law of Cooling
Newton had modeled the rate of change in temperature of a cooled or heated object as proportional to the difference between the object and the surrounding medium.

Example. What happens to the temperature of a cup of soup when it is placed on a table in a room? What does a typical solution curve look like?

Let $H(t)$ be the soup's temperature (in degrees) with respect to time (and we assume that $H(t)$ is a differentiable function).
Let H_s be the constant surrounding temperature.
According to Newton'd law of cooling:

$$\frac{dH}{dt} = -k(H - H_s), \quad k > 0$$

Why $-k$? Because the temperature of the soup is decreasing, the rate of change has to be negative.

What is the equilibrium value? $H = H_s$

$$H'' = -kH' = k^2(H - H_s)$$

$$\begin{array}{ccc} H' > 0 & & H' < 0 \\ H'' < 0 & & H'' > 0 \\ \longrightarrow & & \longleftarrow \\ & H = H_s & \end{array}$$