Math/Stat 2300 Modeling with Differential Equations
from text *Single Variable Calculus*, Stewart and from text *A First Course in Mathematical Modeling*, Giordano, Fox, Horton, Weir, 2009.

Population Growth (11.1 in text)

Law of Natural Growth

A particular model of human population was proposed in the eighteenth century by Thomas Malthus. He assumed that the human population would grow exponentially. His model is what we now consider as the law of natural growth.

One way of modeling populations is to make that assumption that the population grows at a rate proportional to the size of the population

$$\frac{dP}{dt} = kP$$

If k is positive, then the population increases.
If k is negative, then the population decreases.

The differential equation

$$\frac{dP}{dt} = kP$$

is a separable differential equation, which we can solve as follows by integration

$$\begin{aligned} \int \frac{dP}{P} &= \int k\,dt \\ \ln|P| &= kt + C \\ |P| &= e^{kt+C} = e^C e^{kt} \\ P &= Ae^{kt} \end{aligned}$$

What does A represent?

$$A \text{ is the initial value } A = P(0) = P_0$$

Another way of writing the differential equation is as

$$\frac{1}{P}\frac{dP}{dt} = k$$

This says that the **relative growth rate** (the growth rate divided by the population size) is constant. From the solution, we see that this implies that a population with a constant relative growth rate grows exponentially.

How would we verify our model?

$$\text{plotting } \ln\left(\frac{P}{P_0}\right) \text{ vs. } t$$

Example The following table give the midyear population of Japan, in thousands, from 1960 to 2005:

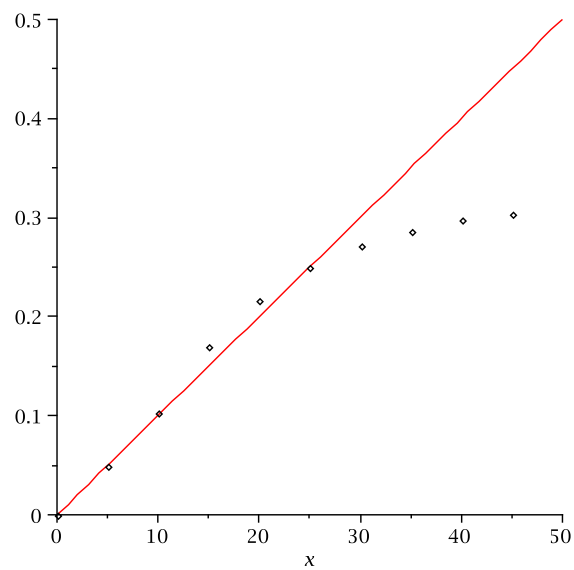
Year	Population
1960	94 092
1965	98 883
1970	104 345
1975	111 573
1980	116 807
1985	120 754
1990	123 537
1995	125 341
2000	126 700
2005	127 417

We want to fit this exponential model to this data. First, we can note that we can treat 1960 at $t = 0$ and then just plot t from 0 to 45.
What is P_0? $P_0 = 94092$

Next, to see if the exponential model suits the data, we plot $\ln\left(\frac{P}{P_0}\right)$ vs t, to see if we can fit a straight line to the data. This graph is to the right.

This straight line goes through the origin and has slope $k = 0.01$. Is this a good fit? What does this suggest?

good fit for small P



The Logistic Model

As discussed before, many populations do not grow exponentially, but eventually levels off as it approaches the carrying capacity.
So we consider the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

When P is small, the relative growth rate is almost constant, but as P increases, the relative growth rate decreases and becomes negative if P exceeds the carrying capacity K.

We can also solve this differential equation. First, we rewrite it as

$$\frac{dP}{P\left(1 - \frac{P}{K}\right)} = k$$

$$\Longrightarrow K\frac{dP}{P\left(K - P\right)} = k$$

Using partial fractions (try this one your own)

we obtain

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}$$

Thus, integrating, we have

$$\begin{aligned}
\int \left(\frac{1}{P} + \frac{1}{K-P}\right)\, dP &= \int k\, dt \\
\ln|P| - \ln|K-P| &= kt + C \\
\ln\left|\frac{K-P}{P}\right| &= -kt - C \\
\left|\frac{K-P}{P}\right| &= e^{-kt}e^{-C} \\
\frac{K-P}{P} &= Ae^{-kt}
\end{aligned}$$

which gives

$$P = \frac{K}{1 + Ae^{-kt}}$$

What is A here?

$$A = \frac{K - P_0}{P_0}$$

How can we verify our model?

plot $\ln\left(\frac{P}{K-P}\right)$ vs. t

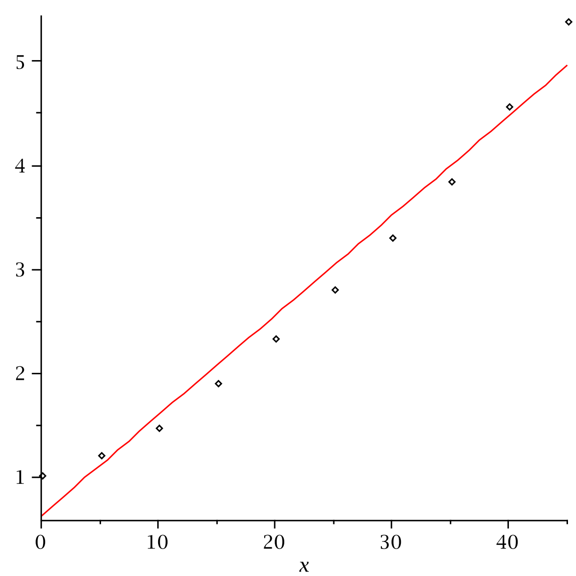
Example (continuation of previous example with Japan's population)

We want to fit this logistic model to this data. Again, we can treat 1960 at $t = 0$ and then just plot t from 0 to 45.

From the data, what is an estimate for K?

128000

To see if the logistic model suits the data (and to estimate our parameters), we plot $\ln\left(\frac{P}{K-P}\right)$ vs t, to see if we can fit a straight line to the data. This graph is to the right.



The fitted straight line has slope 0.9615 with intercept 0.6308.

Do you think this is a good fit? What is our model?

$$P = \frac{128000}{1 + 0.532e^{-0.9615t}}$$