Math/Stat 2300 Final Review

Practice Questions Solutions

1. Cubic splines are different cubic polynomials between successive data points that interpolates the data. We solve for the coefficients of these cubics by:
   - the splines must satisfy the data points (it is interpolation)
   - the first and second derivatives must match at each interior data point
   - another two independent equations: conditions on the derivatives at the endpoints.

2. Consider the differential equation

$$\frac{dP}{dt} = r(M - P)P$$

   (a) Setting the left-hand side to zero:

$$r(M - P)P = 0 \Longrightarrow P = 0 \text{ or } M - P = 0 \Longrightarrow P = M$$

   The equilibrium values are $P = M$ and $P = 0$.

   (b)

$$\frac{d^2P}{dt^2} = rM - 2rPP' = rM - 2rP(r(M - P)P) = rM - 2r^2P^2(M - P)$$

   (c) The phase line diagram is as follows. Notice that the second derivative has been included to see the concavity of the solution curves.

$$\begin{array}{ccccccc}
P'' < 0 & & P'' > 0 & & P'' < 0 & & P'' > 0 \\
P' < 0 & & P' > 0 & & P' > 0 & & P' < 0 \\
\longleftarrow & & \longrightarrow & & \longrightarrow & & \longleftarrow \\
\hline
& P = 0 & & P = \frac{M}{2} & & P = M &
\end{array}$$

3. Separate the variables and integrate:

$$\int \frac{dy}{y} = \int 2t + 3t^2 \, dt \Longrightarrow y = Ae^{t^2+t^3}$$

   Then, using the initial value,

$$1 = Ae^{(0)^2+(0)^3} \Longrightarrow A = 1$$

   The solution is

$$y = e^{t^2+t^3}$$

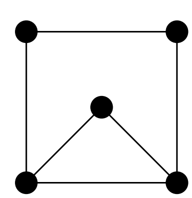
4. Here $P(x) = 2$ and $Q(x) = e^{-x}$. Then
$$\mu(x) = e^{\int P(x)dx} = e^{2x}$$
Then
$$\int \mu(x)Q(x)\ dx = \int e^{2x}e^{-x}\ dx = e^x + C$$
Then, combining everything, the general solution is
$$\mu(x)y = \int \mu(x)Q(x)\ dx + C$$
$$y = \frac{e^x + C}{e^{2x}}$$

5. (a) So $x_0 = 0$, $y_0 = 1$. Let $f(x,y) = (1+y)x$ The x-values are:
$$x_1 = x_0 + \Delta x = 0.1, x_2 = 0.2, x_3 = 0.3$$
Then the approximations are
$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)\Delta x = 1 + (1+1)(0)(0.1) = 1 \\ y_2 &= y_1 + f(x_1, y_1)\Delta x = 1 + (1+1)(0.1)(0.1) = 1.02 \\ y_3 &= y_2 + f(x_2, y_2)\Delta x = 1.2 + (1+1.02)(0.2)(0.1) = 1.0404 \end{aligned}$$
(b) Reducing the step size will increase the accuracy of the approximations.

6. Describe the graph associated with a particular social network. What do the nodes represent? What do the edges represent?

(Many different possibilities, any of the ones discussed in class are fine) The Bacon number graph where the nodes are actors or actresses and the edges are movies.

7. (a)
$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
(b) (There may be multiple appropriate graphs. All should have the same number of edges and vertices with the same degree)



8. (a) Dijkstra's algorithm finds the shortest path from one particular vertex in a weighted graph to all other vertices.

(b) Kruskal's algorithm find a minimum spanning tree of a weighted graph.

9. (a) Give an example of a problem that could be modeled using simulation.

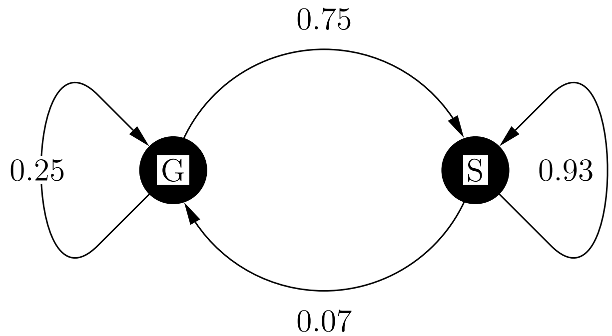
    (Many possibilities) One example are inventory models (where demand and other variables are random) where

    (b) Monte Carlo simulation involves repeated trials using random numbers.

    From a given probability distribution, we divide up $[0, 1]$ with each interval having size corresponding to the probability for each possible outcome. We generate a random number between 0 and 1, simulating the random variable by using the above intervals.

10. Consider a model for the long-term dining behaviour of the students at a particular college. It is found that 25% of the students who eat at the college's Grease Dining Hall return to eat there again, whereas those who eat at Sweet Dining Hall have a 93% return rate. These are the only two dining halls available on campus, and assume that all students eat at one of these dining halls.

    (a) Let S represents the Sweet Dining Hall. Let G represents the Grease Dining Hall.



    (b) Here the variables are defined as in the part (b)

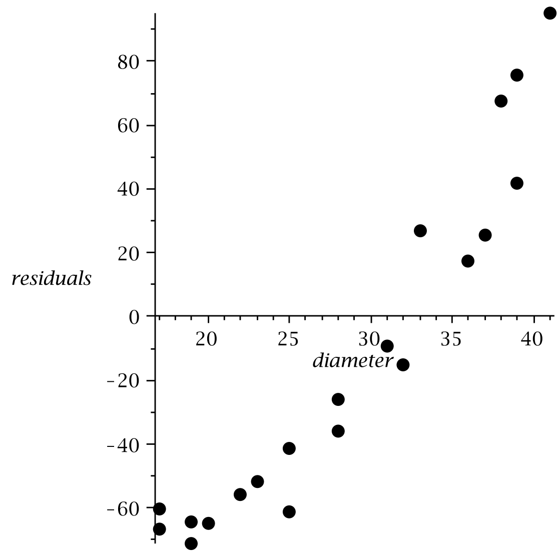
	S	G
S	0.93	0.07
G	0.75	0.25

    (c) Let G_n be the number of people who eat at the Grease Dining Hall in period n. Let S_n be the number of people who eat at the Sweet Dining Hall in period n.

$$
\begin{aligned}
a_{n+1} &= 0.25a_n + 0.07b_n \\
b_{n+1} &= 0.75a_n + 0.93b_n
\end{aligned}
$$

11. (a) When fitting data, we would use linear regression if the data consisted of multiple observation points y_i for each value of the independent variable x_i.

(b) Consider the following plot of residuals vs the independent variable x:



A pattern or trend in the residuals indicates that some effect or aspect on the data has not been captured in the model. Here, it appears that the residuals have a power function pattern (a quadratic or a cubic). The model can then be refined by adding a quadratic or cubic term.