Math/Stat 2300 Modeling using Graph Theory (March 23/25)
from text *A First Course in Mathematical Modeling*, Giordano, Fox, Horton, Weir, 2009.

Describing Graphs (8.2)

A **graph** is a mathematical way of describing relationships between things.
A graph consists of vertices and edges. A graph G consists of two sets: a **vertex set** $V(G)$ and an **edge set** $E(G)$. Each element of $E(G)$ is a pair of elements from $V(G)$.

When an edge ij has a vertex j as one of its endpoints, we say edge ij is **incident** with vertex j. When there is an edge ij between two vertices, we say vertices i and j are **adjacent**.

The **degree** of a vertex is the number of edges incident to that vertex.
A **directed graph** is a graph where the edges have direction, that is, instead of edges we have arrows.
The **indegree** of a vertex of a directed graph is the number of arrows pointing to a vertex.
The **outdegree** of a vertex of a directed graph is the number of arrows pointing out of a vertex.

Graph Models (8.3)

The Seven Bridges of Königsberg

Problem: Can the seven bridges be traversed exactly once starting and ending at the same place?

Graph Coloring

Problem: Given a geographic map, is it possible to colour it with four colours so that any two regions that share a common border are assigned different colours?

This problem can alternatively be framed as:
Problem: Using only four colours, can you colour the vertices of a planar graph so that no vertex gets the same colour as an adjacent vertex?

Traveling Salesman Problem

Problem: A salesman must visit every location in a list and return to the starting location. In what order should the salesman visit the locations in the list to minimize the distance traveled?

This problem can be modeled using a graph, where the locations on the list are the vertices and the edges are the paths between each location. Here each edge has a *cost* c_{ij} which can be distance or cost of travel. A *tour* is a list of all locations, in any order.

Restated in graph theory terms:

Traveling Salesman Problem Given a complete graph $G(V(G), E(G))$ and a cost c_{ij} for each edge in $E(G)$, find a tour of minimum cost.

Policing a City (Vertex Cover Problem)

Suppose we want to organize a company of soldiers to police all the roads in a section of a city. The section of the city can be modeled with a graph where the road intersections in the city are vertices and the roads between the intersections are edges. Suppose the roads in the city are straight enough and short enough so that a soldier stationed at an intersection can effectively police all roads immediately incident with that intersection.

Problem: What is the smallest number of soldiers needed to police all the roads?

Vertex Cover Problem: Given a graph $G = (V(G), E(G))$, the vertex cover problem seeks a subset $S \subseteq V(G)$ of the smallest number of elements such that every edge in the graph is incident with at least one elements in S.

Fitting a Piecewise Linear Function to Data

Suppose we have some set of data that we want to fit using straight line segments, but that we don't necessarily need as many line segments as we have data points. One way of solving this is to consider a simple graph problem.

Web graph

The World-Wide Web can be represented as a graph where the nodes are webpages and the links are directed edges. Search engines treat the world-wide web as a graph and index the websites based on the indegree.

Social Networks

A **social network** consists of a set of individuals, groups, or organizations with relationships between them. These can be modeled with a graph.

Some examples:

- Bacon number graph

  A popular game is the "Six Degrees of Kevin Bacon" where you connect a particular actor to the actor Kevin Bacon by using a small number of connections. Two actors are "connected" if they both appeared in the same movie.

  For example,

  Val Kilmer was in the movie "Top Gun" with Tom Cruise who was in "A Few Good Men" with Kevin Bacon.

  The Bacon number graph would be the graph with the vertices are actors and the edges are movie connections. The Bacon number for an actor would be the shortest number of connections between the particular actor and Kevin Bacon.

- Collaboration graph

  A collaboration graph is a graph with vertices are authors of papers and the edges are papers written together.

- Erdös number graph

  A particular example of a collaboration graph is the Erdös number graph. Paul Erdös is a prolific mathematician. A person's Erdös number is the "collaborative distance" between that person and Paul Erdös, that is, the shortest number of connections between the author and Erdös.

  For example, Richard Nowakowski wrote a paper with Richard Guy who wrote a paper with Paul Erdös. So Richard Nowakowski has Erdös number of 2.

- friendship network

  A friendship network is a graph where the vertices are the people and an edge represents friendship between two people.

Adjacency Matrix

An *adjacency matrix* of a graph is a matrix where the entries represents edges between vertices, that is, a '1' represents that vertices are adjacent to each other.

The matrix for the above example is:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Using Graph Models to Solve Problems (8.4)

We have examined real-world problems that can be expressed as problems on graphs. Now, we talk about methods for solving some of these graph theory problems.

Solving Shortest-Path Problems

Given a graph G, edge lengths c_{ij} for each edge $ij \in E(G)$, and two specific vertices u and v, the shortest path problem is compute the length of a shortest path from u to v.

Dijkstra's Shortest Path Algorithm

Input A graph $G = (V(G), E(G))$ with a source vertex s and a sink vertex t and nonnegative edge lengths c_{ij} for each edge $ij \in E(G)$.

Output The length of a shortest path from s to t in G.

Step 0 Start with temporary labels L on each vertex as follows: $L(s) = 0$ and $L(i) = \infty$ for all vertices except s.

Step 1 Find the vertex with the smallest temporary label (if there's a tie, pick one at random). Make that label permanent meaning it will never change.

Step 2 For every vertex j without a permanent label that is adjacent to a vertex with a permanent label, compute a new temporary label as follows: $L(j) = \min\{L(i) + c_{ij}\}$ where we minimize over all vertices i with a permanent label. Repeat steps 1 and 2 until all vertices have permanent labels.

Example. Consider the following graph:

We will use Dijkstra's algorithm to find the shortest distance from vertex u to every other vertex.

We will use the notation for the labels $L(V) = (L(u), L(v), L(w), L(x), L(y), L(z))$. We will use an asterisk to not that a label we have made permanent.

Step 0 We initialize the labels as
$L(V) = (L(u), L(v), L(w), L(x), L(y), L(z)) = (0, \infty, \infty, \infty, \infty, \infty)$.