Math/Stat 2300 Modeling using Graph Theory - Part II (March 23/25)
from text *A First Course in Mathematical Modeling*, Giordano, Fox, Horton, Weir, 2009.

## Adjacency Matrix

An *adjacency matrix* of a graph is a matrix where the entries represents edges between vertices, that is, a '1' represents that vertices are adjacent to each other.

*Formal definition.* Given a graph $G$ with $n$ vertices labeled $v_1, v_2, \ldots, v_n$. For each $i$ and $j$ $(1 \leq i \leq n, 1 \leq j \leq n)$, we define

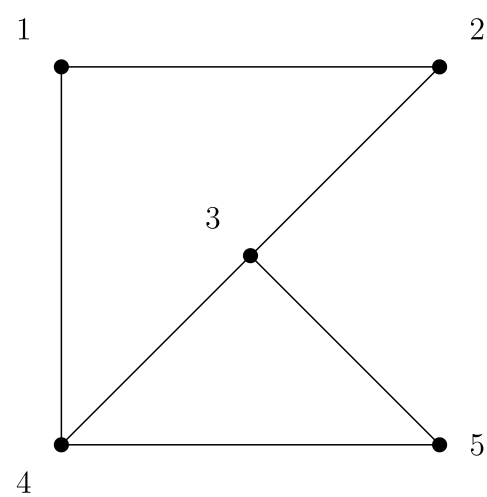
$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an edge} \\ 0 & \text{if } v_i v_j \text{ is not an edge} \end{cases}$$

The *adjacency matrix* of $G$ is the $n \times n$ matrix $A = [a_{ij}]$ whose $(i, j)$ entry is $a_{ij}$.

*Properties of an Adjacency Matrix.* Let $G$ be a graph with vertices $v_1, v_2, \ldots, v_n$ and let $A = [a_{ij}]$ be the adjacency matrix of $G$.

- The diagonal entries are all 0.
- The matrix is symmetric, that is, $a_{ij} = a_{ji}$ for all $i, j$.

  Given a symmetric matrix $A$ which contains only 0's and 1's and only 0's along its diagonal, there exists a graph $G$ whose adjacency matrix is $A$.
- deg $v_i$ is the number of 1's in row $i$
- The $(i, j)$ entry of $A^2$ is the number of different walks from $v_i$ to $v_j$ of length 2. The $(i, j)$ entry of $A^n$ is the number of different walks from $v_i$ to $v_j$ of length $n$.




The matrix for the above example is:

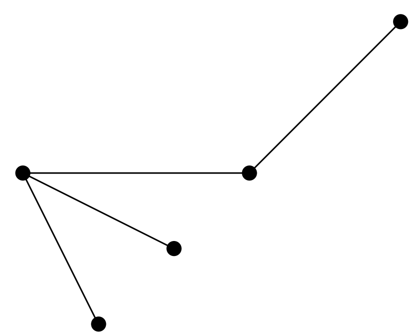
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Then we have the following:

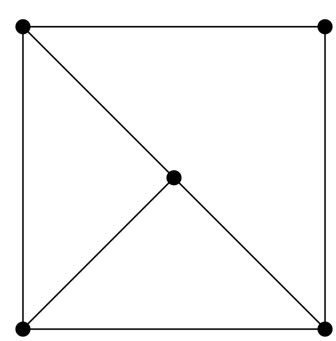
$$A^2 = \begin{bmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 2 & 0 & 3 & 1 & 1 \\ 0 & 2 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 4 & 1 & 5 & 2 \\ 4 & 0 & 5 & 1 & 2 \\ 1 & 5 & 2 & 6 & 4 \\ 5 & 1 & 6 & 2 & 4 \\ 2 & 2 & 4 & 4 & 2 \end{bmatrix}$$

## Spanning Trees

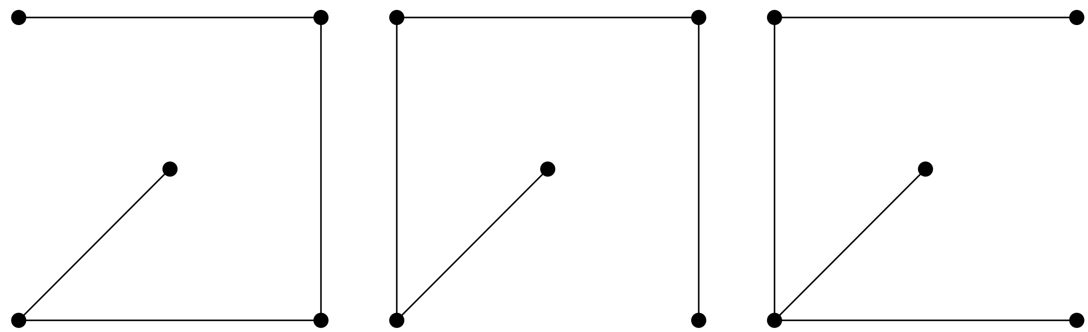
*Definitions.*
A **tree** is a graph where any two vertices are connected by exactly one simple path. It is a graph without any cycles.



A **spanning tree** of a connected graph $G$ is a subgraph which is a tree and which includes every vertex of $G$.
Consider the following graph:



Three of spanning trees for this graph are:



A **minimum spanning tree** of a weighted graph is a spanning tree of least weight, that is, a spanning tree for which the sum of the weights of all its edges is least among all spanning trees.

*Kruskal's Algorithm.*
This algorithm finds a minimum spanning tree in a connected weighted graph with $n > 1$ vertices.

**Step 1**. Find an edge of least weight and call this $e_1$. Set $k = 1$.

**Step 2**. **While** $k < n$

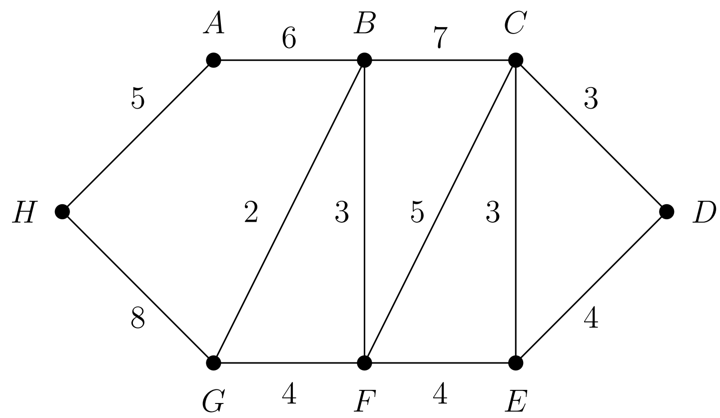
if there exists an edge $e$ such that $\{e\} \bigcup \{e_1, e_2, \ldots, e_k\}$ does not contain a circuit (cycle); let $e_{k+1}$ be such an edge of least weight and replace $k$ by $k + 1$

else output $e_1, e_2, \ldots, e_k$ and stop

**end while**

We can use this same algorithm to find a maximum spanning tree, by find the edge of most weight instead of least weight in the algorithm.

*Example.* Consider the following graph. Find a minimum spanning tree.




*Applications.*

*References for extra reading.*


- *Discrete Mathematics with Graph Theory*, E.G. Goodaire and M.M. Parmenter, Prentice Hall, 2005.
- *Introduction to Graph Theory*, D. West, Upper Saddle River, NJ: Prentice Hall, 2001.