Math/Stat 2300 Piecewise Polynomial Approximation
from text *A First Course in Mathematical Modeling*, Giordano, Fox, Horton, Weir, 2009.

Linear Splines

Linear splines are straight lines between successive data points. Although it can be helpful, it is essentially just "connects the dots".
Consider three data points (x_1, y_1), (x_2, y_2) and (x_3, y_3). When x is in the interval $x_1 \le x \le x_2$, we construct a linear spline $S_1(x)$ passing through the data points (x_1, y_1) and (x_2, y_2):

$$S_1(x) = a_1 x + b_1, \quad x_1 \le x \le x_2$$

Similarly, when $x_2 \le x \le x_3$, we construct a linear spline $S_2(x)$ passing through the data points (x_2, y_2) and (x_3, y_3):

$$S_2(x) = a_2 x + b_2, \quad x_2 \le x \le x_3$$

Each spline is uniquely defined (two points define a line) and the splines meet at (x_2, y_2). The difficulty with linear splines is that the model then lacks smoothness. Although the splines meet at (x_2, y_2), the derivatives do not agree there.

Cubic Splines(4.4)

Cubic spline interpolation is a technique that fits different cubic polynomials between successive pairs of data points. This technique incorporates smoothness into the model by requiring that the first **and** second derivatives of adjacent splines agree at each data point. Similar to linear splines, we define the splines as follows: Consider three data points (x_1, y_1), (x_2, y_2) and (x_3, y_3). When x is in the interval $x_1 \le x \le x_2$, we define the spline

$$S_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1$$

When x is in the interval $x_2 \le x \le x_3$, we define the spline

$$S_2(x) = a_2 x^3 + b_2 x^2 + c_2 x + d_2$$

First, we want the spline to satisfy the data points and to agree at the point $x = x_2$. Therefore

$$\begin{aligned}
y_1 &= a_1 x_1^3 + b_1 x_1^2 + c_1 x_1 + d_1 \\
y_2 &= a_1 x_2^3 + b_1 x_2^2 + c_1 x_2 + d_1 \\
y_2 &= a_2 x_2^3 + b_2 x_2^2 + c_2 x_2 + d_2 \\
y_3 &= a_2 x_3^3 + b_2 x_3^2 + c_2 x_3 + d_2
\end{aligned}$$

Next, we note that there are eight unknowns and four equations. We need other equations to determine the remaining unknown constants. To do this, we first match the first derivative and the second derivative at x_2:

$$S_1'(x_2) = 3a_1 x_2^2 + 2b_1 x_2 + c_1 = 3a_2 x_2^2 + 2b_2 x_2 + c_2 = S_2'(x_2)$$

and

$$S_1''(x_2) = 6a_1x_2 + 2b_1 = 6a_2x_2 + 2b_2 = S_2''(x_2)$$

Now, for the final conditions, we say something about the endpoints. There are multiple ways to do this.
One way is to require that there be no change in the first derivative at the exterior endpoints:

$$S_1''(x_1) = 6a_1x_1 + 2b_1 = 0$$
$$S_2''(x_3) = 6a_2x_3 + 2b_2 = 0$$

We call a spline determined in this way a **natural spline**.
Alternatively, if the values of the derivative at the exterior endpoints are known, then we require that the derivatives match these known values. For example, if the derivatives at the endpoints are given as $f'(x_1)$ and $f'(x_3)$, then

$$S_1'(x_1) = 3a_1x_1^2 + 2b_1x_1 + c_1 = f'(x_1)$$
$$S_2'(x_3) = 3a_2x_3^2 + 2b_2x_3 + c_2 = f'(x_3)$$

We call a spline determined in this way a **clamped spline**.

Example For the following data, we construct natural cubic splines that pass through the data points.

x_i	0	1	2
y_i	0	5	8

Let $S_1(x)$ be the spline

$$S_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1, \quad 0 \leq x \leq 1$$

and let $S_2(x)$ be the spline

$$S_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2, \quad 1 \leq x \leq 2$$

$$\begin{aligned}
0 &= a_1(0)^3 + b_1(0)^2 + c_1(0) + d_1 && (1)\\
5 &= a_1(1)^3 + b_1(1)^2 + c_1(1) + d_1 && (2)\\
5 &= a_2(1)^3 + b_2(1)^2 + c_2(1) + d_2 && (3)\\
8 &= a_2(2)^3 + b_2(2)^2 + c_2(2) + d_2 && (4)\\
S_1'(x_2) = S_2'(x_2) &\implies 3a_1 + 2b_1 + c_1 = 3a_2 + 2b_2 + c_2 && (5)\\
S_1''(x_2) = S_2''(x_2) &\implies 6a_1 + 2b_1 = 6a_2 + 2b_2 && (6)\\
S_1''(x_1) = 0 &\implies 6a_1x_1 + 2b_1 = 0 && (7)\\
S_2''(x_3) = 0 &\implies 6a_2x_3 + 2b_2 = 0 && (8)
\end{aligned}$$

Solving the resulting system of equations, we obtain the constants:

$$a_1 = \frac{3}{11}, a_2 = -\frac{3}{11}, b_1 = 0, b_2 = \frac{18}{11}, c_1 = \frac{52}{11}, c_2 = \frac{34}{11}, d_1 = 0, d_2 = \frac{6}{11}$$