Math 2300 Assignment #4

due by 11:59pm March 25, 2010

Please include your Maple worksheet with your assignment. You may email your Maple worksheet to me. If you do email, please print to a .pdf file and email me a .pdf document.

1. *Learning Curves.* Psychologists interested in learning theory study learning curves. A learning curve is the graph of a function $P(t)$, the performance of someone learning a skill as a function of the training time t. The derivative $\frac{dP}{dt}$ represents the rate at which the performance improves.

   (a) When do you think P increases most rapidly? What happens to $\frac{dP}{dt}$ as t increases? Explain.

   (b) Consider the model
   $$\frac{dP}{dt} = k(M - P), \quad k > 0, M > 0$$
   where M is the maximum level of performance of which the learner is capable. Is this differential equation a reasonable model for learning? Explain.

   (c) Find any equilibrium points of the differential equation given in part (b) and determine their stability.

   (d) Draw a phase-line diagram. Using the phase line diagram, sketch a few solutions curves.

2. A *Bernoulli differential equation* is one of the form
   $$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

   (a) If $n = 0$ or $n = 1$, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation
   $$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

   (b) Using the method in part (a), solve the differential equation
   $$y' + \frac{2}{x}y = \frac{y^3}{x^2}$$

3. (a) Solve the initial value problem

$$\frac{dy}{dx} = 2xy + 2y, \quad y(0) = 3$$

(b) In Maple, create a procedure for Euler's method. It should allow you to input x_0 and y_0, the number of steps n and the step size Δx. It should output the x_i values and the approximate solution values y_i. Recall Euler's method is as follows:

$$y_{i+1} := y_i + f(x_i, y_i)\Delta x$$

Using this, find the first 5 approximations of the given initial value problem with a stepsize $\Delta x = 0.5$. Output the points (x_i, y_i).

(c) Using your procedure from part (b), solve for the first 5 approximations of the given initial value problem with stepsize $\Delta x = 0.1$. Output the points (x_i, y_i).

(d) Using your Maple procedure from part (b), generate a numerical solution for the $\Delta x = 0.5$ and $\Delta x = 0.1$ on the interval $x = [0, 1]$. Plot both numerical solutions and the exact solution from part (a) on the same axis.

4. *Mixing problems.* A typical mixing problem involves a tank of fixed capacity filled with a thoroughly mixed solution of some substance. A solution of a given concentration enters the tank at a fixed rate and the mixture (after being thoroughly mixed) leaves at a fixed rate. If $y(t)$ denotes the amount of substance in the tank at time t, then $y'(t)$ is the rate at which the substance being added minus the rate at which it is being removed. (Similar formulations can result in models for chemical reactions, injection of a drug into the bloodstream, and pollutants into a lake).

(a) Consider a specific mixing problem: a tank contains 100L of water. A brine solution (water saturated with salt) that has a concentration of 0.4 kg/L is added at a rate of 5L/min. The solution is kept mixed and is drained from the tank **at the same rate**. If $y(t)$ is the amount of salt (in kilograms) after t minutes, determine and solve the separable differential equation. How much salt remains in the tank after an hour? (*Watch out for your units!*)

(b) If the rates of flow into and out of the system are different, the volume is not constant.

A tank contains 100L of water. A brine solution (water saturated with salt) that has a concentration of 0.4 kg/L is added at a rate of 5L/min. The solution is kept mixed and is drained from the tank at a rate of 3L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, determine and solve the linear differential equation.