Math/Stat 2300 Midterm Review Partial Solutions

1.
$$a_n = (0.9)^n a_0$$

$$a = 0.9a \Longrightarrow a = 0$$

Let $f(a) = 0.9a$, $f'(0) = 0.9 < 1$, so $a = 0$ is an stable fixed points.

2.
$$a = 0.8a + 100 \Longrightarrow a = 500$$

$$f(a) = 0.8a + 100 \Longrightarrow f' = 0.8 \Longrightarrow f'(500) = 0.8 < 1$$

So $a = 500$ is stable.

3. There is no equilibrium here.

4. Let S_n be the number of students infected with the flu after n time periods. Then

$$S_{n+1} = S_n + \Delta S_n = S_n + kS_n(150 - S_n)$$

k is the constant of proportionality. It represents the number of interactions that develop into the flu (where the number of interactions is $S_n(150 - S_n)$), or you could just consider it as the rate of which the flu spreads.

6. (a) Given a set of data points (x_i, y_i), $i = 1, 2, \ldots, m$, fitting the data to the curve $y = f(x)$.

   Applying the Least Squares criterion is the problem of determining the parameters of the function type $f(x)$ to minimize that sum of the squares of the absolute deviations, that is,

$$\sum_{i=1} m|y_i - f(x_i)|^2$$

   (b) Applying Least Squares criterion to fit a straight line, we have the following equations

$$\begin{aligned} a &= \frac{m\sum x_i y_i - \sum x_i \sum y_i}{m\sum x_i^2 - \left(\sum x_i\right)^2} \\ b &= \frac{m\sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i}{m\sum x_i^2 - \left(\sum x_i\right)^2} \end{aligned}$$

   Using Least Squares, fit the following data to a straight line:

x	0	1	2	3
y	2	6	7	9

$$\begin{aligned} a &= \frac{4(47)-(6)(24)}{4(14)-36} = \frac{44}{20} = 2.2 \\ b &= \frac{(14)(24)-(47)(6)}{4(14)-36} = \frac{54}{20} = 2.7 \end{aligned}$$

The line that minimizes the sum of squared deviations is $2.2x + 2.7$.

7. Consider the data:

x	0	1	2	3
y	1	2	7	22

(a)

$$P(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x)$$

$$\begin{aligned} L_0(x) &= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}(x-1)(x-2)(x-3) \\ L_1(x) &= \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}x(x-2)(x-3) \\ L_2(x) &= \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = -\frac{1}{2}x(x-1)(x-3) \\ L_3(x) &= \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{1}{6}x(x-1)(x-2) \end{aligned}$$

$$P(x) = -\frac{1}{6}(x-1)(x-2)(x-3)(1) + \frac{1}{2}x(x-2)(x-3)(2) - \frac{1}{2}x(x-1)(x-3)(7) + \frac{1}{6}x(x-1)(x-2)(22)$$

(b) divided difference table

Data		Divided Differences		
x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	1			
		$\frac{2-1}{1-0} = 1$		
1	2		$\frac{5-1}{2-0} = 2$	
		$\frac{7-2}{2-1} = 5$		$\frac{5-2}{3-0} = 1$
2	7		$\frac{15-5}{3-1} = 5$	
		$\frac{22-7}{3-2} = 15$		
3	22			

$$P_3(x) = 1 + x + 2x(x-1) + x(x-1)(x-2)$$

8. (a) $y = Ae^{bx}$

   We would plot $\ln(y)$ vs. x. If it was a straight line, the slope of the line would give b and the intercept would give $\ln(A)$.

   (b) $y = Ax^2$

   We would plot y vs. x^2. If it was a straight line through the origin, the slope of the line would give A.

   (c) $y = Ae^{x^2}$

   We would plot $\ln(y)$ vs x^2. If it was a straight line with slope 1, the intercept would give $\ln(A)$.

   Note: there may be alternative solutions.

9. x is not proportional to $\ln(y)$ because it is not a straight line through the origin. The model is

$$y = 1.284e^{0.75x}$$

   Note: there may be alternate models.

5. Integrating

$$P = -Ae^{-kt} - M$$

   k could be considered as a relative growth rate. It is a proportionality constant. M is the number to which the population will tend. It is a carrying capacity.