Math/Stat 2300 Modeling using Graph Theory (March 23/25)
from text *A First Course in Mathematical Modeling*, Giordano, Fox, Horton, Weir, 2009.

Separation of Variables

When we had

$$\frac{dy}{dx} = x(1-y),$$

we called it a separable equation and we wrote it as

$$\frac{dy}{1-y} = x\ dx$$

and integrated. We were using separation of variables.
Consider the differential equation

$$\frac{dy}{dx} = f(x,y).$$

If $f(x,y)$ is a function of x multiplied by a function of y, that is,

$$f(x,y) = p(x)q(x)$$

we can solve the differential equation using **separation of variables**.

Example. Solve $y' = 4x^3e^{-y}$.

Separate the variables:

$$e^y\ dy = 4x^3\ dx$$

Integrate

$$\int e^y\ dy = \int 4x^3\ dx \Longrightarrow e^y = x^4 + C$$

Apply the natural logarithm

$$y = \ln(x^4 + C)$$

How do we verify that this is a solution?

Differentiate

$$\begin{aligned} y' &= \frac{1}{x^4+C}4x^3 \\ &= 4x^3(x^4+C)^{-1} \\ &= 4x^3e^{-\ln(x^4+C)} = 4x^3e^{-y} \end{aligned}$$

Example. Solve $\frac{dy}{dx} = 2t(1+y^2)$.

Integrate

$$\int \frac{dy}{1+y^2} = \int 2t\ dt$$

Then

$$\arctan(y) = t^2 + C \Longrightarrow y = \tan(t^2 + C)$$

Verify.

$$\begin{aligned} y' &= \sec^2(t^2+C)\cdot 2t \\ &= (1+\tan^2(t^2+C))\cdot 2t \\ &= (1+y^2)\cdot 2t \\ &= 2t(1+y^2) \end{aligned}$$

Linear Equations (11.7)

First-order linear equation is an equation of the form

$$a_1(x)y' + a_0(x)y = b(x)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x.
Division gives the **standard form** of the linear equation

$$y' + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous on an interval.

Consider the case where $P(x)$ is constant and $Q(x)$ is zero.

$$y' + ky = 0$$

where k is a constant. This gives the separable equation

$$y' + ky = 0 \Longrightarrow \frac{dy}{dx} = -k$$

Integrating

$$y = Ce^{-kx}$$

Consider the general linear equation

$$y' + P(x)y = Q(x)$$

Let $\mu(x)$ be some function of x.

$$\mu(x)[y' + P(x)y] = \mu(x)[Q(x)]$$

Consider the left-hand side

$$\mu(x)[y' + P(x)y]$$

Let's say that $\mu(x)$ must satisfy

$$\mu(x)[y' + P(x)y] = \frac{d}{dx}[\mu(x)y] = \mu(x)y' + \mu'(x)y$$

What must $\mu(x)$ be? The equation gives

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)y' + \mu'(x)y$$

So, we must have

$$\mu(x)P(x) = \mu'(x)$$

Integrating and solving for $\mu(x)$

$$\mu(x) = e^{\int P(x)dx}$$

We call $\mu(x)$ an integrating factor. So now we have

$$\mu(x)[y' + P(x)y] = \mu(x)Q(x) \Longrightarrow \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

Integrate both sides

$$\mu(x)y = \int \mu(x)Q(x)\ dx + C$$

Then

$$y = \frac{\int \mu(x)Q(x)\ dx + C}{\mu(x)}$$

Example. Find the general solution of

$$xy' + y = e^x, \quad x > 0$$

Step 1: write in standard form:

$$y' + \left(\frac{1}{x}\right) y = \left(\frac{1}{x}\right) e^x$$

So $P(x) = \frac{1}{x}$ and $Q(x) = \frac{e^x}{x}$

Step 2: calculate integrating factor

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

Step 3: Multiply the right-hand side of equation by $\mu(x)$ and integrate

$$\int \mu(x)Q(x)\ dx = \int x\frac{e^x}{x}\ dx = \int e^x\ dx = e^x + C$$

Step 4: Write the general solution$\mu(x)y = \int \mu(x)Q(x)\ dx + C$

$$xy = e^x + C \Longrightarrow y = \frac{e^x + C}{x}$$

Verify.
Differentiate

$$y' = -\frac{1}{x^2}(e^x + C) + \frac{1}{x}e^x$$

Then

$$
\begin{aligned}
xy' + y &= \left[-\frac{1}{x}(e^x + C) + e^x\right] + \left(\frac{e^x + C}{x}\right) \\
&= \frac{-(e^x + C)}{x} + e^x + \left(\frac{e^x + C}{x}\right) \\
&= e^x
\end{aligned}
$$

Uniqueness of Solutions

Theorem (Existence and Uniqueness) Suppose that $P(x)$ and $Q(x)$ are continuous functions over the interval $\alpha < x < \beta$. Then there is one and only one function $y = y(x)$ satisfying the first order linear equation

$$y' + P(x)y = Q(x)$$

on the interval and the initial condition

$$y(x_0) = y_0$$

at the specified point x_0 in the interval.

Example. Solve the initial value problem

$$xy' + 2y = x^2, \ y(1) = 1$$

We rewrite in standard form:

$$y' + \frac{1}{x}y = x$$

Here $P(x) = \frac{2}{x}$ and $Q(x) = x$.
First, we determine the integrating factor

$$\mu(x) = e^{\int P(x)dx} = e^{2\ln(x)} = x^2$$

Then

$$\int \mu(x)Q(x)\ dx = \int x^3\ dx = \frac{1}{4}x^4 + C$$

So, we have

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

Now, we solve for C using our initial condition.

$$1 = \frac{1}{4} + C \Longrightarrow C = \frac{3}{4}$$

Then

$$y = \frac{1}{4}x^2 + \frac{3}{4}x^{-2}$$