Math 1000 Extra Problems Solutions

Related Rates

P1. If a (spherical) snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

The surface area of the snowball is $A = 4\pi r^2$ (where r is a radius of the snowball). Let D be the diameter, and note that $r = \frac{D}{2}$. Then $A = \pi D^2$.

We are given that $\frac{dA}{dt} = -1$ cm^2/min. Differentiating,

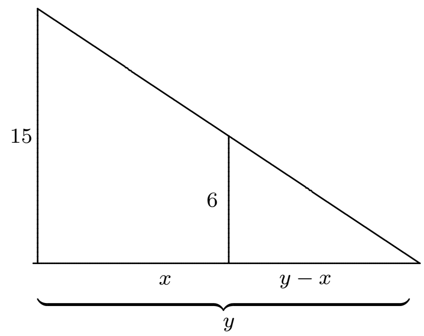
$$\frac{dA}{dt} = 2\pi D \frac{dD}{dt}$$

When $D = 10$, we solve for $\frac{dD}{dt}$:

$$\frac{dD}{dt} = \frac{1}{2\pi D}\frac{dA}{dt} \Longrightarrow \frac{dD}{dt} = \frac{1}{2\pi(10)}(-1) \approx -0.0159$$

So the diameter decreases at 0.0159 cm/min.

P2. A man 6 feet tall walks at a rate of 5 feet per second away from a street light that is 15 feet above the ground. When he is 10 feet away from the base of the light, at which rate is the tip of his shadow moving?



We are given that $\frac{dx}{dt} = 5$ ft/sec. We want to find $\frac{dy}{dt}$ when $x = 10$.

By similar triangles,

$$\frac{y}{15} = \frac{y-x}{6} \Longrightarrow y = \frac{5}{3}x$$

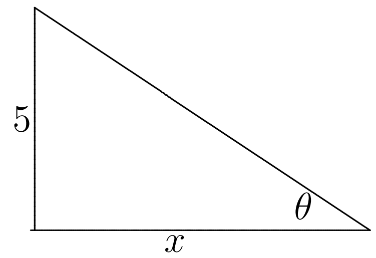
Differentiating,

$$\frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$$

Thus,

$$\frac{dy}{dt} = \frac{5}{3}(5) = \frac{25}{3} \text{ ft/sec.}$$

P3. An airplane flies at an altitude of 5 miles toward a point directly over an observer. The speed of the plane is 600 miles per hour. Find the rate at which the angle of elevation θ is changing when the angle is $\frac{\pi}{4}$.



Let x be the distance traveled by the plane. Then $\frac{dx}{dt} = 600$ mi/hr. So

$$\tan(\theta) = \frac{5}{x}$$

Differentiating,

$$\begin{aligned} \sec^2(\theta)\frac{d\theta}{dt} &= -5x^{-2}\frac{dx}{dt} \\ \frac{d\theta}{dt} &= -5(\cos(\theta))^2 x^{-2}\frac{dx}{dt} \end{aligned}$$

When $\theta = \frac{\pi}{4}$, $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ and

$$\tan(\frac{\pi}{4}) = \frac{5}{x} \Longrightarrow 1 = \frac{5}{x} \Longrightarrow x = 5$$

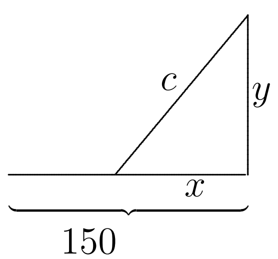
.

Thus,

$$\frac{d\theta}{dt} = -5(\frac{1}{\sqrt{2}})^2(5)^{-2}(600) = -60$$

.

P4. At noon, Ship A is 150 km west of Ship B. Ship A is travelling east at 35 km/hr and Ship B is travelling north at 25 km/hr. How fast is the distance between the ships changing at 4:00pm?



Let y be the distance traveled by Ship B and let $150 - x$ be the distance Ship A has traveled. Then $\frac{dy}{dt} = 25$ km/hr and $\frac{dx}{dt} = -35$ km/hr. Let c be the distance between the two ships. Then

$$c^2 = x^2 + y^2$$

Differentiating

$$2c\frac{dc}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dc}{dt} = \frac{1}{c}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

At 4:00pm, four hours have passed. So Ship B has traveled $25 \cdot 4 = 100$ km and Ship A has traveled $35 \cdot 4 = 140$. Thus, at 4:00pm, $y = 100$ and $x = 150 - 140 = 10$ and

$$c = \sqrt{x^2 + y^2} = \sqrt{10{+}100^2} = \sqrt{10100}$$

Therefore,

$$\frac{dc}{dt} = \frac{1}{\sqrt{10100}}((10)(-35) + (100)(25)) = \frac{2150}{\sqrt{10100}} = \frac{215}{\sqrt{101}}$$

Optimization

P5. Find the dimensions of a rectangle with area 100 m^2 whose perimeter is as small as possible.

Let x and y be the dimensions of the rectangle. We are given $xy = 100$ and we want to minimize the perimeter $P = 2x + 2y$.

So $y = \frac{100}{x}$ which implies that $P = 2x + \frac{200}{x}$. Differentiating,

$$P' = 2 - \frac{200}{x^2}$$

$$P' = 0 \Longrightarrow 2 - \frac{200}{x^2} = 0 \Longrightarrow x = \pm\sqrt{100} \Longrightarrow x = \pm 10$$

We only consider the positive value of x since the dimensions of the rectangle can not be negative.

	$0 < x < 10$	$10 < x$
P'	$-$	$+$

By the First Derivative Test, $x = 10$ is a minimum (since P' changes from a negative to a positive at $x = 10$).

The dimensions of the rectangle which minimize perimeter are $x = 10$ and $y = 10$.

P6. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

We want to minimize the distance from the parabola $y = 4 - x^2$ to the point $(0, 2)$. So we want to minimize

$$\begin{aligned} f(x) = d^2 &= (x-0)^2 + ((4-x^2)-2)^2 \\ &= x^2 + (2-x^2)^2 \\ &= x^2 + 4 - 4x^2 + x^4 \\ &= x^4 - 3x^2 + 4 \end{aligned}$$

Differentiating

$$f'(x) = 4x^3 - 6x$$

$$f' = 0 \Longrightarrow 4x^3 - 6x = 0 \Longrightarrow 2x(2x^2 - 3) = 0 \Longrightarrow x = 0, x = \pm\sqrt{\frac{3}{2}}$$

	$x < -\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}} < x < 0$	$0 < x < \sqrt{\frac{3}{2}}$	$x > \sqrt{\frac{3}{2}}$
f'	$-$	$+$	$-$	$+$

By the First Derivative Test, f has a minimum at $x = \pm\sqrt{\frac{3}{2}}$, when f has the value $f(\pm\sqrt{\frac{3}{2}}) = \frac{5}{2}$.

So the points $(\sqrt{\frac{3}{2}}, \frac{5}{2})$ and $(\sqrt{\frac{3}{2}}, \frac{5}{2})$ are closest to the point $(0, 2)$.

P7. Find two positive numbers whose product is 192 and whose sum is a minimum.

Let x and y be our two positive numbers. We are given that $xy = 192$ and we want to minimize $S = x + y$.

So $y = \frac{192}{x}$ which implies that $S = x + \frac{192}{x}$. Differentiating

$$S' = 1 - \frac{192}{x^2}$$

$$S' = 0 \Longrightarrow 1 - \frac{192}{x^2} = 0 \Longrightarrow x = \pm\sqrt{192}$$

We only look at the positive value because both x and y are positive.

	$0 < x < \sqrt{192}$	$\sqrt{192} < x$
S'	$-$	$+$

By the First Derivative Test, $x = \sqrt{192}$ makes S a minimum.

$$x = \sqrt{192} \Longrightarrow y = \frac{192}{\sqrt{192}} = \sqrt{192}$$

Therefore, the two positive numbers are both $\sqrt{192}$.

Definite Integrals

P8.

$$\int_1^e \frac{1}{x}\,dx = \ln(x)\big|_1^e = \ln(e) - \ln(1) = 1$$

P9.

$$\int_1^2 (x^2-3)\,dx = \frac{x^3}{3} - 3x\bigg|_1^2 = \frac{2^3}{3} - 3(2) - \left(\frac{1^3}{3} - 3(1)\right) = \frac{-2}{3}$$

P10.

$$\int_{-\pi/2}^{\pi/2} (2t+\cos(t))\,dt = t^2 + \sin(t)\big|_{-\pi/2}^{\pi/2} = \left(\frac{\pi}{2}\right)^2 + \sin(\frac{\pi}{2}) - \left(\left(\frac{-\pi}{2}\right)^2 + \sin(\frac{-\pi}{2})\right) = 2$$

P11.

$$\int_\pi^\pi \sin(x)\,dx = 0$$

Indefinite Integrals

P12.

$$\int (4x^3 + \frac{1}{x^2})\,dx = 4\frac{1}{4}x^4 + \frac{x^{-1}}{-1} + C = x^4 - \frac{1}{x} + C$$

P13.

$$\int x(x^2+3)\,dx = \int x^3 + 3x\,dx = \frac{1}{4}x^4 + \frac{3}{2}x^2 + C$$

P14.

$$\int (\sec^2(\theta) - \sin(\theta))\,d\theta = \tan(\theta) + \cos(\theta) + C$$

P15.

$$\int (\cos(x) + 3^x)\,dx = \sin(x) + \frac{3^x}{\ln(x)} + C$$

Integrals using Substitution

P16. Let $u = 5x$. Then $du = 5dx$.

$$\int 5e^{5x}\,dx = \int e^u\,du = e^u + C = e^{5x} + C$$

P17. Let $u = x^2+1$. Then $du = 2xdx \Longrightarrow xdx = \frac{1}{2}du$

$$\int x(x^2+1)^2\,dx = \int u^2\frac{1}{2}du = \frac{1}{2}\frac{1}{3}u^3 + C = \frac{1}{6}(x^2+1)^3 + C$$

P18. Let $u = \sin(3x)$. Then $du = 3\cos(3x)dx \Longrightarrow \cos(3x)dx = \frac{1}{3}du$

$$\int \sin^2(3x)\cos(3x)\ dx = \int u^2 \frac{1}{3}du = \frac{1}{3}\frac{1}{3}u^3 + C = \frac{1}{9}(\sin(3x))^3 + C$$

P19. [typo in this question. It should read:

$$\int_1^5 \frac{x}{\sqrt{2x^2-1}}\ dx.]$$

Let $u = 2x^2 - 1$. Then $du = 4xdx \Longrightarrow xdx = \frac{1}{4}du$. Also,

$$x = 1 \Longrightarrow u = 2(1)^2 - 1 = 1$$

$$x = 5 \Longrightarrow u = 2(5)^2 - 1 = 49$$

$$\int_1^5 \frac{x}{\sqrt{2x^2-1}}\ dx = \int_1^{49} \frac{1}{\sqrt{u}}\frac{1}{4}du = \frac{1}{2}\sqrt{u}\Big|_1^{49} = 3$$

P20. Let $u = 1 - x$. Then $du = -dx \Longrightarrow dx = -du$. Also,

$$x = 1 \Longrightarrow u = 1 - 1 = 0$$

$$x = 2 \Longrightarrow u = 1 - 2 = -1$$

$$\int_1^2 e^{1-x}\ dx = \int_0^{-1} -e^u du = \int_{-1}^0 e^u du = e^u|_{-1}^0 = 1 - \frac{1}{e}$$

P21. Let $u = 2x$. Then $du = 2dx \Longrightarrow dx = \frac{1}{2}du$. Also,

$$x = 0 \Longrightarrow u = 2(0) = 0$$

$$x = \frac{\pi}{2} \Longrightarrow u = 2(\frac{\pi}{2}) = \pi$$

$$\int_0^{\pi/2} \sin(2x)\ dx = \int_0^{\pi} \sin(u)\frac{1}{2}du = \frac{-1}{2}\cos(u)\Big|_0^{\pi} = 1$$