Math 1000 Final Exam Review Solutions

1. Consider $f(x) = \begin{cases} -2x+3 & \text{if } x < 0 \\ x^2 & x \geq 0 \end{cases}$. Find the following limits:

   (a) $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2 = 0$

   (b) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} -2x+3 = 3$

   (c) $\lim\limits_{x\to 0} f(x)$ does not exist.

2. Find each of the following limits:

   (a) $\lim\limits_{x\to -3} \dfrac{x^2+x-6}{x+3} = \lim\limits_{x\to -3} \dfrac{(x+3)(x-2)}{x+3} = \lim\limits_{x\to -3} x-2 = -3-2 = -5$

   (b)

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}\frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{x}{x(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

   (c)

$$\lim_{x\to 0} \frac{\sin(4x)}{\tan(3x)} = \lim_{x\to 0} = \frac{\sin(4x)}{\frac{\sin(3x)}{\cos(3x)}} = \lim_{x\to 0} \frac{\sin(4x)}{\sin(3x)}\cos(3x)\frac{3x}{3x}\frac{4x}{4x}$$

$$= \lim_{x\to 0} \frac{\sin(4x)}{4x}\frac{3x}{\sin(3x)}\cos(3x)\frac{4x}{3x} = \lim_{x\to 0} \frac{\sin(4x)}{4x}\frac{3x}{\sin(3x)}\cos(3x)\frac{4}{3} = \frac{4}{3}$$

3. (a) Give the limit definition of the derivative.

   The derivative of a function f at a number a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

   (b) Use the **limit definition of derivative** to find $f'(1)$ if $f(x) = \sqrt{x+3}$.

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1+h+3}-\sqrt{1+3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}\frac{(\sqrt{4+h}+2)}{(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{h}{h(\sqrt{4+h}+2)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}
\end{aligned}$$

4. (a) State the Mean Value Theorem.

   Let f be a function that satisfies the following conditions: (i) f is continuous on the closed interval $[a, b]$ and (ii) f is differentiable on the open interval (a, b). Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   (b) State Rolle's Theorem.

   Let f be a function that satisfies the following conditions: (i) f is continuous on the closed interval $[a, b]$, (ii) f is differentiable on the open interval (a, b), and (iii) $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$.

   (c) Let $f(x) = x^2$ on the interval $[-2, 1]$. Show that the Mean Value Theorem applies and then find all the values of c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   Since f is a polynomial, it is continuous on $[-2, 1]$ and it is differentiable on $(-2, 1)$. Thus, the Mean Value Theorem applies.

   Differentiating $f'(x) = 2x$. By the Mean Value Theorem, we know that

$$\begin{aligned} f'(c) = 2c &= \frac{f(1) - f(-2)}{1 - (-2)} \\ &= \frac{(1)^2 - (-2)^2}{1 - (-2)} \\ &= \frac{1 - 4}{3} = -1 \\ c &= \frac{-1}{2} \end{aligned}$$

5. Differentiate the following functions:

   (a) $H(x) = xe^x + x^2 \cos(ax) + \arcsin(x)$, where a is a constant

$$H'(x) = xe^x + e^x + 2x\cos(ax) - ax^2 \sin(ax) + \frac{1}{\sqrt{1 - x^2}}$$

   (b) $f(t) = \dfrac{t^2 - 1}{t^2 + 1}$

$$\begin{aligned} f' &= \frac{(t^2 + 1)(2t) - (t^2 - 1)(2t)}{(t^2 + 1)^2} \\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2 + 1)^2} \\ &= \frac{4t}{(t^2 + 1)^2} \end{aligned}$$

(c) $g(x) = \arctan\left(\sqrt{x+e^x}\right)$

$$\begin{aligned} g' &= \frac{1}{1+(\sqrt{x+e^x})^2}\frac{1}{2}(x+e^x)^{-1/2}(1+e^x) \\ &= \frac{1+e^x}{2\sqrt{x+e^x}(1+x+e^x)} \end{aligned}$$

(d) $f(x) = \ln(\sin(2x^2))$

$$f'(x) = \frac{1}{\sin(2x^2)}\cos(2x^2)(4x) = \frac{4x\cos(2x^2)}{\sin(2x^2)}$$

6. (a) Differentiate implicitly to find the derivative of y if

$$\sqrt{x} + \sqrt{y} = 3.$$

$$\begin{aligned} \frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' &= 0 \\ \frac{1}{2}y^{-1/2}y' &= \frac{-1}{2}x^{-1/2} \\ y' &= \frac{-x^{-1/2}}{y^{-1/2}} = \frac{-\sqrt{y}}{\sqrt{x}} \end{aligned}$$

(b) Find the equation of the tangent line of the function in part (a) at the point $(4,1)$.
The slope of the tangent line at the point $(4,1)$ is

$$y' = \frac{-\sqrt{1}}{\sqrt{4}} = \frac{-1}{2}$$

Then the equation of the tangent line is

$$y - 1 = \frac{-1}{2}(x-4) \Longrightarrow y = \frac{-1}{2}x + 3.$$

7. Differentiate with respect to x (use logarithmic differentiation):

(a) $y = \dfrac{xe^x(x-1)^{3/2}}{\sqrt{x+1}}$

First, take the natural log of both sides

$$\begin{aligned} \ln(y) &= \ln\left(\frac{xe^x(x-1)^{3/2}}{\sqrt{x+1}}\right) \\ &= \ln(xe^x(x-1)^{3/2}) - \ln(\sqrt{x+1}) \\ &= \ln(x) + \ln(e^x) + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) \\ &= \ln(x) + x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) \end{aligned}$$

Differentiating implicitly, we obtain

$$\begin{aligned}
\frac{1}{y}y' &= \frac{1}{x} + 1 + \frac{3}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1} \\
y' &= y\left(\frac{1}{x} + 1 + \frac{3}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}\right) \\
&= \frac{xe^x(x-1)^{3/2}}{\sqrt{x+1}}\left(\frac{1}{x} + 1 + \frac{3}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}\right)
\end{aligned}$$

(b) $y = (1+x)^{1/x}$

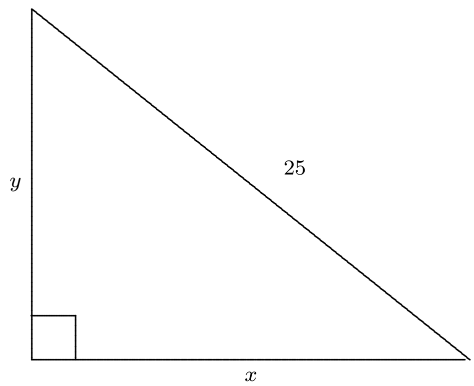
Take the natural log of both sides

$$\ln(y) = \frac{1}{x}\ln(1+x)$$

Differentiating implicitly, we obtain

$$\begin{aligned}
\frac{1}{y}y' &= (-1)x^{-2}\ln(1+x) + \frac{1}{x}\frac{1}{1+x} \\
y' &= y\left(\frac{-\ln(1+x)}{x^2} + \frac{1}{x(1+x)}\right) \\
&= (1+x)^{1/x}\left(\frac{-\ln(1+x)}{x^2} + \frac{1}{x(1+x)}\right)
\end{aligned}$$

8. A ladder (24 feet long) is leaning against the wall of a house. The bottom of the ladder is sliding away from the wall at a rate of 2 feet/sec. How was is the top of the ladder moving down the wall, when the bottom of the ladder is 24 feet away from the wall?



Let x be the distance that the bottom of the ladder is away from the wall. Let y be the distance the top of the ladder is from the ground. So we are given that $\frac{dx}{dt} = 2$ ft/sec and we want to find $\frac{dy}{dt}$ when $x = 24$.

Since the ladder makes a right-angled triangle with the wall, we have

$$x^2 + y^2 = 25.$$

When $x = 24$, $y = \sqrt{25^2 - 24^2} = \sqrt{(25-24)(25+24)} = \sqrt{49} = 7$.

Differentiating our equation with respect to t, we obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solving for $\frac{dy}{dt}$,

$$\begin{aligned} 2y\frac{dy}{dt} &= -2x\frac{dx}{dt} \\ \frac{dy}{dt} &= \frac{-x}{y}\frac{dx}{dt} \end{aligned}$$

So, when $x = 24$, $y = 7$ and

$$\frac{dy}{dt} = \frac{-(24)}{7}(2) = \frac{-48}{7} \approx -6.8$$

9. Find the linearization $L(x)$ of $f(x) = x^4 + 3x^2$ for $a = -1$.

Recall the formula for the linearization of f: $L(x) = f(a) + f'(a)(x - a)$.

$$\begin{aligned} f(a) &= f(-1) = (-1)^4 + 3(-1)^2 = 1 + 3 = 4 \\ f'(x) &= 4x^3 + 6x \\ f'(a) &= f'(-1) = 4(-1)^3 + 6(-1) = -4 - 6 = -10 \end{aligned}$$

Then

$$L(x) = 4 + (-10)(x - (-1)) \Longrightarrow L(x) = -10x - 6.$$

10. Find the following limits:

(a)

$$\begin{aligned} &\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} && \frac{0}{0} \text{ form} \\ &\overset{H}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} && \frac{0}{0} \text{ form} \\ &\overset{H}{=} \lim_{x\to 0} \frac{e^x}{2} \\ &= \frac{1}{2} \end{aligned}$$

(b) $\lim_{x\to\infty} x^{1/x}$

Let $y = \lim_{x\to\infty} x^{1/x}$. Then, taking the natural log,

$$\ln(y) = \lim_{x\to\infty} \ln(x^{1/x}) = \lim_{x\to\infty} \frac{1}{x}\ln(x)$$

Then this limit has the form $\frac{\infty}{\infty}$. We use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Therefore,

$$y = \lim_{x\to\infty} x^{1/x} = e^0 = 1.$$

11. Given the function $f(x) = 2 + 3x - x^3$, answer the following:

(a) What is $f'(x)$?

$$f' = 3 - 3x^2$$

(b) Find the critical numbers.

Set f' to zero and solve for x:

$$\begin{aligned} 3 - 3x^2 &= 0 \\ x^2 &= 0 \\ x &= \pm 1 \end{aligned}$$

The critical numbers are $x = 1$ and $x = -1$.

(c) Find the intervals of increasing/decreasing.

	$x < -1$	$-1 < x < 1$	$x > 1$
f'	$-$	$+$	$-$

$f' < 0$ on $(-\infty, -1)$ and $(1, \infty)$ so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

$f' > 0$ on $(-1, 1)$ so f is increasing on $(-1, 1)$.

(d) Find the local maximum and/or the local minimum value.

f changes from increasing to decreasing at $x = 1$, so f has a maximum value of $f(1) = 4$.

f changes from decreasing to increasing at $x = -1$, so f has a minimum value of $f(-1) = 0$.

(e) Find the intervals of concavity.

First, we find the second derivative: $f'' = -6x$. Setting f'' to zero and solving for x, we get $x = 0$.

$f'' > 0$ for $x < 0$, so f is concave up on $(-\infty, 0)$.

$f'' < 0$ for $x > 0$, so f is concave down on $(0, \infty)$.

(f) What is the inflection point?

Since f changes concavity at $x = 0$, $(0, f(0)) = (0, 2)$ is an inflection point.

12. Given the function $f(x) = \frac{x-1}{x^2}$, answer the following:

(a) What is the domain?

When $x = 0$, the function is undefined. So our domain is all x in the real numbers such that $x \neq 0$. We can write this as $D = (-\infty, 0) \bigcup (0, \infty)$.

(b) $x = 0$ is not in the domain, so there are no y-intercepts. Setting $y = 0$, we get that $x = 1$ is an intercept. Therefore, $(1, 0)$ is an x-intercept.

(c) Are there any vertical asymptotes? Are there any horizontal asymptotes? If so, what are they?

$\lim_{x\to 0} \dfrac{x-1}{x^2} = -\infty$, so $x = 0$ is a vertical asymptote.

$$\lim_{x\to\infty} \frac{x-1}{x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1} = 0.$$

Thus, $y = 0$ is a horizontal asymptote.

$$\lim_{x\to-\infty} \frac{x-1}{x^2} \cdot \frac{1/x^2}{1/x^2} = \lim_{x\to-\infty} \frac{\frac{1}{x} - \frac{1}{x^2}}{1} = 0.$$

This just gives the same horizontal asymptote.

(d) Find the critical numbers.

Taking the derivative:

$$\begin{aligned} f' &= \frac{(1)(x^2) - (x-1)(2x)}{(x^2)^2} \\ &= \frac{x^2 - (2x^2 - 2x)}{x^4} = \frac{x^2 - 2x^2 + 2x}{x^4} \\ &= \frac{2x - x^2}{x^4} = \frac{x(2-x)}{x^4} \\ &= \frac{2-x}{x^3} \end{aligned}$$

Set $f' = 0$: $x = 2$

Therefore, $x = 2$ is a critical number. $x = 0$ is not a critical number because it is not in the domain.

(e) Find the intervals of increasing/decreasing.

	$x < 0$	$0 < x < 2$	$x > 2$
$2 - x$	$+$	$+$	$-$
x^3	$-$	$+$	$+$
f'	$-$	$+$	$-$

$f' < 0$ on $(-\infty, 0)$ and $(2, \infty)$, so f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

$f' > 0$ on $(0, 2)$, so f is increasing on $(0, 2)$.

(f) What are the local maximum value and the local minimum value?

Since f changes from increasing to decreasing at $x = 2$, f has a maximum value of $f(2) = \frac{1}{4}$. f does not have any maxima or minima at $x = 0$ since f has a vertical asymptote at $x = 0$.

(g) Find the intervals of concavity.

First, we find the second derivative:

$$\begin{aligned}
f'' &= \frac{(-1)(x^3) - (2-x)(3x^2)}{(x^3)^2} \\
&= \frac{-x^3 - (6x^2 - 3x^3)}{x^6} \\
&= \frac{2x^3 - 6x^2}{x^6} = \frac{2x^2(x-3)}{x^6} \\
&= \frac{2(x-3)}{x^4}
\end{aligned}$$

Setting f'' to zero and solving for x, we get $x = 3$.

	$x < 0$	$0 < x < 3$	$x > 3$
$2(x-3)$	$-$	$-$	$+$
x^4	$+$	$+$	$+$
f'	$-$	$-$	$+$

Since $f'' < 0$ on $(-\infty, 0)$ and $(0, 3)$, f is concave down on $(-\infty, 0)$ and $(0, 3)$.

Since $f'' > 0$ on $(3, \infty)$, f is concave up on $(3, \infty)$.

(h) Find the inflection points.

Since f changes concavity at $x = 3$, $(3, f(3)) = (3, \frac{2}{9})$ is an inflection point.

(i) Using parts (a)-(g), graph the function.

13. Find two numbers whose difference is 100 and whose product is a minimum.

Let our two numbers be x and y. We are given that $y - x = 100$ and we want to minimize their product: $P = xy$.

First, $y = 100 + x$. Then we rewrite $P = x(100 + x) = 100x + x^2$. Differentiating

$$P' = 100 + 2x$$

Setting to zero and solving for x, we obtain

$$P' = 0 \Longrightarrow 100 + 2x = 0 \Longrightarrow x = -50$$

Taking the second derivative, we have $P'' = 2 > 0$. Since $P'(-50) = 0$ and $P'' > 0$, by the Second Derivative Test, P has a minimum at $x = -50$. Then, $y = 100 + x = 100 + (-50) = 50$. So our two numbers are -50 and 50.

14. (a) Estimate the area under the curve of $f(x) = 4 - x^2$ on the interval $[0, 2]$ using 4 approximating rectangles and right endpoints.

$\Delta x = \frac{2-0}{4} = \frac{1}{2}$

$$\begin{aligned} R_4 &= \frac{1}{2}f(\frac{1}{2}) + \frac{1}{2}f(1) + \frac{1}{2}f(\frac{3}{2}) + \frac{1}{2}f(2) \\ &= \frac{1}{2}\left(4 - (\frac{1}{2})^2 + 4 - (1)^2 + 4 - (\frac{3}{2})^2 + 4 - (2)^2\right) \\ &= \frac{1}{2}\left(4 - \frac{1}{4} + 4 - 1 + 4 - \frac{9}{4}\right) \\ &= \frac{1}{2}\frac{34}{4} = \frac{17}{4} \end{aligned}$$

(b) State the definition of the definite integral of f from a to b (as the limit of a Riemann sum).

The **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_i^* are the sample points.

(c) Write $\int_0^2 (4 - x^2)\,dx$ as the limit of the Riemann sum over n intervals. (Do not evaluate the limit).

First, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. We can take the sample points to be the right endpoints (it makes no difference to the infinite limit): $x_i = a + i\Delta x = 0 + i\frac{2}{n} = \frac{2i}{n}$.

Then, using the definition from part (b),

$$\int_0^2 (4 - x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 - \left(\frac{2i}{n}\right)^2\right)(\frac{2}{n})$$

15. Evaluate the following integrals

(a)

$$\begin{aligned}
\int_1^9 \frac{x-1}{\sqrt{x}}\,dx &= \int_1^9 \frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\,dx \\
&= \int_1^9 x^{1/2} - x^{-1/2}\,dx \\
&= \left[\frac{x^{3/2}}{3/2} - \frac{x^{1/2}}{1/2}\right]_1^9 = \left[\frac{2}{3}x^{3/2} - 2x^{1/2}\right]_1^9 \\
&= \left(\frac{2}{3}9^{3/2} - 2(9^{1/2})\right) - \left(\frac{2}{3}(1)^{3/2} - 2(1)^{1/2}\right) = \frac{2}{3}3^3 - 2\cdot 3 - \frac{2}{3} + 2 \\
&= \frac{40}{3}
\end{aligned}$$

(b)

$$\int_0^{\pi/2} \cos(x)\,dx = [\sin(x)]_0^{\pi/2} = \sin(\frac{\pi}{2}) - \sin(0) = 1 - 0 = 1$$

(c)

$$\int \sec^2(x)\,dx = \tan(x) + C$$

(d) $\int e^x \sin(e^x)\,dx$

Use a substitution: let $u = e^x$, then $du = e^x dx$. Then

$$\begin{aligned}
\int e^x \sin(e^x)\,dx &= \int \sin(e^x)\,e^x dx \\
&= \int \sin(u)du \\
&= -\cos(u) + C = -\cos(e^x) + C
\end{aligned}$$

(e) $\int_0^7 \sqrt{4+3x}\,dx$

Use a substitution: let $u = 4 + 3x$, then $\frac{1}{3}du = dx$. Also, changing the limits of integration: $x = 0 \Longrightarrow u = 4$ and $x = 7 \Longrightarrow u = 25$. Then

$$\begin{aligned}
\int_0^7 \sqrt{4+3x}\,dx &= \int_4^{25} \frac{1}{3}\sqrt{u}\,du \\
&= \left[\frac{1}{3}\frac{2}{3}x^{3/2}\right]_4^{25} \\
&= \frac{2}{9}(25^{3/2} - 4^{3/2}) \\
&= \frac{2}{9}(5^3 - 2^3) = \frac{2}{9}(117) \\
&= 26
\end{aligned}$$