Math 1000 Midterm Exam Solutions

Monday, July 21, 2008

1. Consider $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if } x < 2 \\ x-1 & \text{if } 2 \le x < 3 \\ \frac{1}{x} & \text{if } x \ge 3 \end{cases}$.

   (6 marks) (a)Find each of the following limits:

   (i)

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-x-2}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2^-} x+1 = 3$$

   (ii)

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x-1 = 1$$

   (iii) $\lim\limits_{x\to 2} f(x)$ does not exist because the r.h.s. and l.h.s. limits are not equal.

   (iv)

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x-1 = 2$$

   (v)

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{1}{x} = \frac{1}{3}$$

   (vi) $\lim\limits_{x\to 3} f(x)$ does not exist because the r.h.s. and l.h.s. limits are not equal.

   (1 mark) (b) Give the intervals over which $f(x)$ is continuous.

   $(-\infty, 2), (2, 3), (3, \infty)$

   or

   $(-\infty, 2), [2, 3), [3, \infty)$

2. Evaluate the following limits:

   (2 marks) (a)

$$\lim_{t\to -3} \frac{t^2-9}{2t^2+7t+3} = \lim_{t\to -3} \frac{(t-3)(t+3)}{(2t+1)(t+3)} = \lim_{t\to -3} \frac{t-3}{2t+1} = \frac{6}{5}$$

   (2 marks) (b)

$$\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2} \cdot \frac{\sqrt{t^2+16}+4}{\sqrt{t^2+16}+4} = \lim_{x\to 0} \frac{x^2+16-16}{x^2(\sqrt{t^2+16}+4)}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{t^2+16}+4)} = \lim_{x\to 0} \frac{1}{(\sqrt{t^2+16}+4)} = \frac{1}{8}$$

(2 marks) (c)

$$\lim_{\theta\to 0}\frac{\tan(3\theta)}{\sin(2\theta)} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{\cos(3\theta)}\cdot\frac{1}{\sin(2\theta)} = \lim_{\theta\to 0}\frac{\sin(3\theta)}{\cos(3\theta)}\cdot\frac{1}{\sin(2\theta)}\frac{3\theta}{3\theta}\frac{1/(2\theta)}{1/(2\theta)}$$

$$= \lim_{\theta\to 0}\frac{\sin(3\theta)}{3\theta}\cdot\frac{1}{\frac{\sin(2\theta)}{2\theta}}\frac{1}{\cos(3\theta)}\frac{3\theta}{2\theta} = \frac{3}{2}$$

3. (1 mark) (a) Give the limit definition of the derivative of a function f at a number a.

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

or

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

(3 marks) (b) Find the equation of the tangent line to the curve $y = (x+3)^2$ at the point $(1,1)$ **using the limit definition of derivative** from above.

$$\begin{aligned}
y'(a) &= \lim_{h\to 0}\frac{(a+h+3)^2-(a+3)^2}{h}\\
&= \lim_{h\to 0}\frac{(a^2+ah+3a+ah+h^2+3h+3a+3h+9)-(a^2+6a+9)}{h}\\
&= \lim_{h\to 0}\frac{2ah+h^2+6h}{h} = \lim_{h\to 0}\frac{h(2a+6+h)}{h}\\
&= 2a+6
\end{aligned}$$

$$y'(1) = 2(1)+6 = 8$$

$$8 = \frac{y-1}{x-1} \Longrightarrow y-1 = 8(x-1) \Longrightarrow y = 8x-7.$$

(4 marks) 4. Use the Intermediate Value Theorem to show that there is a root of

$$2x^3+x^2+2=0$$

in the interval $(-2,-1)$.

Let $f(x) = 2x^3+x^2+2$.

$$\begin{aligned}
f(-2) &= 2(-2)^3+(-2)^2+2 = -10\\
f(-1) &= 2(-1)^3+(-1)^2+2 = 1
\end{aligned}$$

Since $f(-2) < 0 < f(-1)$, by the IVT, there is a root of the given equation in the interval $(-2,-1)$.

5. Differentiate the following functions with respect to x:

(2 marks) (a) $H(x) = x^8 + 3x^2 + \sin(ax)$, where a is a constant.

$$H' = 8x^7 + 6x + a\cos(ax)$$

(2 marks) (b) $f(x) = \frac{1}{2}x^4 e^{x^2}$

$$f' = 2x^3 e^{x^2} + \frac{1}{2}x^4 e^{x^2}(2x) = 2x^3 e^{x^2} + x^5 e^{x^2}$$

(2 marks) (c) $y = \frac{\ln(x)}{x^2}$

$$y' = \frac{\frac{1}{x}x^2 - \ln(x)(2x)}{x^4} = \frac{1 - 2\ln(x)}{x^3}$$

(3 marks) (d) $f(x) = \arctan\left(\frac{x^2-1}{x^2+1}\right)$

$$\begin{aligned} f' &= \frac{1}{1 + \left(\frac{x^2-1}{x^2+1}\right)^2}\left(\frac{(2t)(t^2+1) - (t^2-1)(2t)}{(t^2+1)^2}\right) \\ &= \frac{1}{1 + \left(\frac{x^2-1}{x^2+1}\right)^2}\left(\frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}\right) \\ &= \frac{1}{1 + \left(\frac{x^2-1}{x^2+1}\right)^2}\left(\frac{4t}{(t^2+1)^2}\right) \\ &= \frac{2t}{t^4+1} \end{aligned}$$

(3 marks) (e) $G(x) = \ln(\sin(\sin(x)))$

$$f' = \frac{1}{\sin(\sin(x))}\cos(\sin(x))\cos(x)$$

(3 marks) 6. Find the equation of the tangent line to the curve $y = -\sin(x) + \cos(x)$ at the point $(0, 1)$.

$$y' = -\cos(x) - \sin(x)$$
$$y'(0) = -\cos(0) - \sin(0) = -1$$

Then the equation of the tangent line is

$$-1 = \frac{y-1}{x-0} \Longrightarrow y - 1 = -x \Longrightarrow y = -x + 1.$$

(4 marks) 7. Find y'' if $x^6 + y^6 = 1$.

Using implicit differentiation

$$\begin{aligned} 6x^5 + 6y^5 \cdot y' &= 0 \\ y' &= \frac{-x^5}{y^5} = -x^5 y^{-5} \end{aligned}$$

Then differentiating again

$$\begin{aligned} y'' &= -5x^4 y^{-5} + (-x^5)(-5y^{-6} \cdot y') \\ &= -5x^4 y^{-5} + 5x^5 y^{-6} \left(-x^5 y^{-5}\right) \\ &= -5x^4 y^{-5} - 5x^{10} y^{-11} \end{aligned}$$

8. Use logarithmic differentiation to find the derivative of the function:

(4 marks) (a) $y = (\sin(x))^x$

Take the natural log of both sides

$$\ln(y) = x \ln(\sin(x))$$

Then differentiating implicitly

$$\begin{aligned} \frac{1}{y} y' &= \ln(\sin(x)) + x \frac{1}{\sin(x)} \cos(x) \\ y' &= y \left(\ln(\sin(x)) + \frac{x \cos(x)}{\sin(x)} \right) \\ &= (\sin(x))^x \left(\ln(\sin(x)) + x \cot(x)\right) \end{aligned}$$

(4 marks) (b) $y = \frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}$

Take the natural log of both sides

$$\begin{aligned} \ln(y) &= \ln \left(\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \right) \\ &= \ln(x^2+1)^4 - \ln\left((2x+1)^3(3x-1)^5\right) \\ &= 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \end{aligned}$$

Then differentiating implicitly

$$\begin{aligned} \frac{1}{y} y' &= \frac{4}{x^2+1}(2x) - \frac{3}{2x+1}(2) - \frac{5}{3x-1}(3) \\ y' &= \left(\frac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \right) \left(\frac{8x}{x^2+1} - \frac{6}{2x+1} - \frac{15}{3x-1} \right) \end{aligned}$$

(4 marks) 9. A bacteria culture initially contains 40 cells and grows at a rate proportional to its size. After 2 hours, the population has increased to 320. Find an expression for the number of bacteria after t hours.

We are given that $P(0) = 40$ and $P(2) = 320$ (where $P(t)$ is the population at t hours). Using the equation $P(t) = P(0)e^{kt}$, we solve for k:

$$\begin{aligned} 320 &= 40e^{k(2)} \\ \frac{320}{40} &= e^{2k} \\ 8 &= e^{2k} \\ k &= \frac{\ln(8)}{2} = \frac{3}{2}\ln(2) \end{aligned}$$

Thus,

$$P(t) = 40e^{\frac{3}{2}\ln(2)t} = 40(2^{3/2})^t.$$

(5 marks) 10. A spotlight on the ground shines on a building 10 m away. If a man 2 m tall walks away from the spotlight toward the building at a speed of 2 m/s, how fast is the length of his shadow decreasing when he is 4 m away from the spotlight?

Let x be the distance that the man has travelled from the spotlight and let s be the length of his shadow. Using similar triangles we get the equation to relate x and s as

$$\frac{x}{2} = \frac{10}{s}$$

Solving for s, we obtain

$$s = \frac{20}{x}.$$

Differentiating

$$\begin{aligned} \frac{ds}{dt} &= 20(-1)x^{-2}\frac{dx}{dt} \\ &= -20\frac{1}{4^2}(2) \\ &= \frac{-20}{8} = \frac{-10}{4} \end{aligned}$$

(3 marks) 11. Find the linearization $L(x)$ of the function $f(x) = \ln(x)$ at $a = 1$.

$$f(a) = f(1) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(a) = f'(1) = 1$$

Then, using the formula, $L(x) = f(a) + f'(a)(x - a)$, we obtain

$$L(x) = 0 + 1(x - 1) = x - 1$$

Total Marks: 60

(BONUS!) 12. Find h' in terms of f' and g' given that

$$h(x) = \sqrt{\frac{f(x) + g(x)}{g(x)}}$$

and make any obvious simplifications.

First, rewrite $h(x)$ as

$$h(x) = \sqrt{\frac{f(x)}{g(x)} + 1}$$

Then

$$h'(x) = \frac{1}{2}\left(\frac{f}{g} + 1\right)^{-1/2}\left(\frac{gf' - fg'}{g^2}\right)$$