Math 1000 Midterm Review Solutions

1.Consider $f(t) = \begin{cases} t^2+3 & \text{if } t \le 1 \\ t-1 & \text{if } t > 1 \end{cases}$.

(a)Find each of the following limits:

(i)

$$\lim_{t\to 1^-} f(t) = \lim_{t\to 1^-} t^2+3 = (1)^2+3 = 4$$

(ii)

$$\lim_{t\to 1^+} f(t) = \lim_{t\to 1^+} t-1 = 0$$

(iii)

$$\lim_{t\to 1} f(t)$$

does not exist because the left-hand limit and the right-hand limit does not equal.

(b) Give the intervals over which $f(t)$ is continuous.

$(-\infty, 1]$ and $(1, \infty)$

2. Evaluate the following limits:

(a)

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2$$

(b)

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{(\sqrt{x+2}+3)}{(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7} \frac{x-7}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{1}{(\sqrt{x+2}+3)} = \frac{1}{6}$$

(c)

$$\lim_{x\to 0} 3x\cot(x) = \lim_{x\to 0} \frac{3x\cos(x)}{\sin(x)} = \lim_{x\to 0} \frac{3x\cos(x)}{\sin(x)} \cdot \frac{1/x}{1/x} = \lim_{x\to 0} \frac{3\cos(x)}{\frac{\sin(x)}{x}}$$

$$= \frac{3\lim_{x\to 0}\cos(x)}{\lim_{x\to 0}\frac{\sin(x)}{x}} = \frac{3\cos(0)}{1} = 3$$

3. Give the limit definition of the derivative of a function f at a number a.

The derivative of a function f at a number a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

if this limit exists.

4.Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(1, 1)$ **using the limit definition of derivative** from above.

First, we determine the slope of the tangent line at the point $(1, 1)$

$$y'(1) = \lim_{h\to 0} \frac{y(1+h)-y(1)}{h} = \lim_{h\to 0} \frac{\sqrt{1+h}-\sqrt{1}}{h} = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{(\sqrt{1+h}+1)}{(\sqrt{1+h}+1)}$$

$$= \lim_{h\to 0} \frac{(1+h)-1}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{h}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{1}{(\sqrt{1+h}+1)} = \frac{1}{2}$$

Then the equation of the tangent line is given by

$$y - 1 = \frac{1}{2}(x-1) \Longrightarrow y = \frac{1}{2}x + \frac{1}{2}.$$

5. Find the derivatives of the following functions:
(a) $G(x) = x^8 + 12x^6 - 4x^4 + 10x^3 - 4x + 5$

$$G'(x) = 8x^7 + 72x^5 - 16x^3 + 30x^2 - 4$$

(b) $f(x) = 3xe^x$

$$f'(x) = 3e^x + 3xe^x$$

(c) $f(x) = x^3 e^{3x}$

$$f'(x) = 3x^2e^{3x} + 3x^3e^{3x}$$

(d) $H(t) = \cos(a^3 - t^3)$

$$H'(t) = -\sin(a^3 - t^3) * (-3t^2) = 3t^2 \sin(a^3 - t^3)$$

(e) $f(t) = \frac{2t^2-4t}{2t-6}$

$$f'(t) = \frac{(2t-6)\frac{d}{dx}[2t^2-4t] - (2t^2-4t)\frac{d}{dx}[2t-6]}{(2t-6)^2} = \frac{(2t-6)(4t-4)-(2t^2-4t)2}{(2t-6)^2}$$
$$= \frac{(8t^2-32t+24)-(4t^2-8t)}{(2t-6)^2} = \frac{4t^2-24t+24}{(2t-6)^2} = \frac{2t^2-12t+12}{(t-3)^2}$$

6. Find the equation of the tangent line to the curve $y = \sin(x) - 2\cos(x)$ at the point $(\frac{\pi}{2}, 1)$.
The slope of the tangent line at $(\frac{\pi}{2}, 1)$ is

$$y' = \cos(x) + 2\sin(x) \Longrightarrow y'(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) + 2\sin(\frac{\pi}{2}) = 0 + 2(1) = 2$$

Then the equation of the tangent line is

$$y - 1 = 2(x - \frac{\pi}{2}) \Longrightarrow y = 2x - \pi + 1$$

7. Find dy/dx (by implicit differentiation) of

$$x^2y^2 + xy = 4.$$

$$
\begin{aligned}
2xy^2 + x^2(2yy') + y + xy' &= 0 \\
y'(2x^2y + x) &= -(2xy^2 + y) \\
y' &= \frac{-(2xy^2 + y)}{(2x^2y + x)} \\
&= \frac{-y(2xy + 1)}{x(2xy + 1)} = \frac{-y}{x}
\end{aligned}
$$

8. Find the derivative of the following:
(a) $y = \arcsin(2x + 1)$

$$y' = \frac{1}{\sqrt{1 - (2x + 1)^2}}(2) = \frac{2}{\sqrt{1 - (2x + 1)^2}}$$

(b) $f(t) = \arctan(t^2)$

$$f'(t) = \frac{1}{1 + (t^2)^2}(2t) = \frac{2t}{1 + (t^2)^2}$$

(c) $g(x) = \ln(\sqrt[5]{x})$

$$g'(x) = \frac{1}{\sqrt[5]{x}} \frac{1}{5} x^{-4/5} = \frac{1}{5} \frac{1}{x}$$

Alternatively, we could rewrite the function, using laws of logs, as $g(x)\frac{1}{5}\ln(x)$ and then obtain

$$g'(x) = \frac{1}{5} \frac{1}{x}$$

9.Use logarithmic differentiation to find the derivative of the function:
(a) $y = (2x + 3)^2(5x^2 + x)^4$
Take natural logs of both sides

$$
\begin{aligned}
\ln(y) &= \ln(2x + 3)^2(5x^2 + x)^4 \\
&= \ln(2x + 3)^2 + \ln(5x^2 + x)^4 \\
&= 2\ln(2x + 3) + 4\ln(5x^2 + x)
\end{aligned}
$$

Now, differentiate implicitly

$$
\begin{aligned}
\frac{1}{y} y' &= \frac{2}{2x + 3}(2) + \frac{4}{5x^2 + x}(10x + 1) \\
\frac{1}{y} y' &= \frac{4}{2x + 3} + \frac{40x + 4}{5x^2 + x} \\
y' &= y\left(\frac{4}{2x + 3} + \frac{40x + 4}{5x^2 + x}\right) \\
y' &= \left((2x + 3)^2(5x^2 + x)^4\right)\left(\frac{4}{2x + 3} + \frac{40x + 4}{5x^2 + x}\right)
\end{aligned}
$$

(b) $y = (\tan(x))^{1/x}$
Take natural logs of both sides

$$\begin{aligned} \ln(y) &= \ln(\tan(x))^{1/x} \\ &= \frac{1}{x}\ln(\tan(x)) \end{aligned}$$

Now, differentiate implicitly

$$\begin{aligned} \frac{1}{y}y' &= -x^{-2}\ln(\tan(x)) + x^{-1}\frac{1}{\tan(x)}\sec^2(x) \\ y' &= y\left(-x^{-2}\ln(\tan(x)) + \frac{\sec^2(x)}{x\tan(x)}\right) \\ &= \left((\tan(x))^{1/x}\right)\left(-x^{-2}\ln(\tan(x)) + \frac{\sec^2(x)}{x\tan(x)}\right) \end{aligned}$$

10. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour, the population has increased to 420. Find an expression for the number of bacteria after t hours.
If the population grows at a rate proportional to its size, then

$$\frac{dP}{dt} = kP.$$

From the theorem in the text (Thm. 2, p.234), this has solution

$$P(t) = P(0)e^{kt}.$$

Here, we have $P(0) = 100$. When $t = 1$, $P(1) = 420$. We use this information to find k.

$$420 = 100e^{k(1)} \Longrightarrow 4.2 = e^k \Longrightarrow k = \ln(4.2)$$

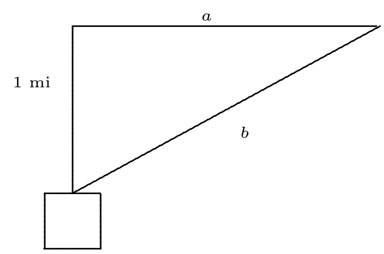
Then

$$P(t) = P(0)e^{kt} = 100e^{\ln(4.2)t}$$

and using exponential laws

$$P(t) = 100(e^{\ln(4.2)})^t = 100(4.2)^t$$

11. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mi/hr passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.



From the information given in the question, we know that $\frac{da}{dt} = 500$ mi/hr. When $b = 2$ miles, we want to determine $\frac{db}{dt}$. We see that the plane's path forms a right triangle, so we have $a^2 + 1^2 = b^2$. When $b = 2$, $a = \sqrt{b^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$. Differentiating, we obtain

$$2a\frac{da}{dt} = 2b\frac{db}{dt}$$

Substituting in the values we know $b = 2$, $\frac{da}{dt} = 500$ and $a = \sqrt{3}$, we get

$$2(\sqrt{3})(500) = 2(2)\frac{db}{dt} \Longrightarrow \frac{db}{dt} = \frac{500\sqrt{3}}{2} = 250\sqrt{3} \text{ mi/hr}$$

Therefore, the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station is $250\sqrt{3}$ mi/hr.
12. Verify the linear approximation

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

at $a = 1$.
We determine the linear approximation of $f(x) = \sqrt{x+3}$ at $a = 1$ and verify that it is equal to what is given. The formula for linear approximation is given as

$$L(x) = f(a) + f'(a)(x-a).$$

Here,

$$f'(x) = \frac{1}{2}(x+3)^{\frac{-1}{2}} \Longrightarrow f'(a) = f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

Also, $f(a) = f(1) = \sqrt{1+3} = 2$. Then we obtain

$$L(x) = 2 + \frac{1}{4}(x-1) = \frac{1}{4}x + 2 - \frac{1}{4} = \frac{1}{4}x + \frac{7}{4}.$$

13. Use the Intermediate Value Theorem to show that there is a root of

$$\cos(x) = x$$

in the interval $(0, \frac{\pi}{2})$.
First, let $f(x) = \cos(x) - x$. Note that $f(x)$ is continuous (since it is the difference of two continuous functions). We have $f(0) = \cos(0) - 0 = 1 > 0$ and $f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2} < 0$. By the Intermediate Value Theorem, there exists a number c in the interval $(0, \frac{\pi}{2})$ where $f(c) = 0$. That means that there is a root of the equation $\cos(x) = x$ in the given interval.