Research Interests: applied analysis, asymptotic analysis, numerical analysis, ordinary and partial differential equations, perturbation methods, pattern formation, mathematical biology and mathematical modelling.

Past and Current Research

My research consists of considering the formation of patterns (localized structures) in reaction-diffusion systems and their stability. Recently, my focus has been on bistable patterns that consist of sharp interfaces that form box-like structures as seen in Figure 1, which are called “mesas”. These types of localized structures are demonstrated in several types of systems and applications. Some such systems are the model of the Belousov-Zhabotinkii reaction in water-in-oil microemulsion ([1], [2], [3]), the Brusselator model ([4], [5]), the Gierer-Meinhardt model with saturation used to model patterns on animal hides ([6], [7], [8]), the Lengyel-Epstein model of the CIMA reaction ([10], [11]), the model of the coexistence of competing species ([12]), the model of vegetation patterns on arid land ([13]) and many others.

The focus of my PhD research was on techniques for examining the formation and stability of these structures.

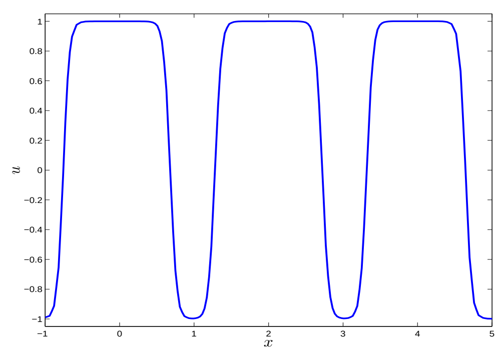

Figure 1: The profile of u forming a mesa pattern

Instability thresholds and dynamics of mesa patterns in one spatial dimension

Consider the reaction-diffusion system

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(u,w) \\ 0 = D w_{xx} + g(u,w) \end{cases} \tag{1}$$

We examine structures such as those seen in Figure 1, in the asymptotic regime where ε is very small and D is exponentially large in the above system. For some fixed D, the system has a steady state that consists of a periodic mesa pattern. In [14], we derive thresholds for D where, as we increase D, the pattern becomes unstable. As well as deriving these thresholds, we consider how the interfaces move in time. We derive equations to describe the dynamics of the interfaces.

In Figure 2, we see that for a two mesa pattern, for this fixed D, the pattern is unstable in time. As the pattern breaks down, the left-hand mesa loses mass to the right-hand mesa and forms one mesa, which, for this D value is stable.

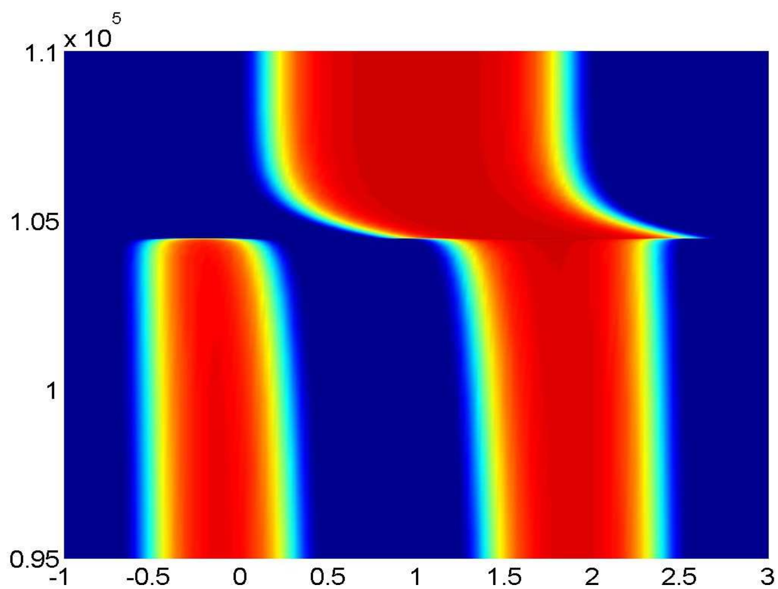

Figure 2: An example of coarsening: two unstable mesas degenerate to one mesa (time vs x, $x \in [-1, 3]$; red is $u = 1$ and blue is $u = -1$)

Mesa patterns on a thin domain

Consider the analogous 2D system by considering the equivalent one dimensional problem

$$\begin{cases} u_t = \frac{\varepsilon^2}{h(x)}[h(x)u_x]_x + f(u,w) \\ \tau w_t = \frac{D}{h(x)}[h(x)w_x]_x + g(u,w) \end{cases}, \tag{2}$$

where $h(x)$ is a positive function corresponding to the domain height. We again examine the stability of mesa patterns. Here the instability comes from effects of the heterogeneous domain. Similar thresholds are determined except now these depend on the function $h(x)$. This work makes up Chapter 3 of my PhD thesis.

Oscillation of mesa patterns

Consider the following system

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + f(u,w) \\ \tau w_t = D w_{xx} + g(u,w) \end{cases} \tag{3}$$

under similar conditions as before. For some values of τ, the solution u consisting of the sharp interfaces oscillates. This oscillation can be seen in Figure 3. Over time for a particular τ, this oscillating solution converges to a constant amplitude as seen in Figure 4. This occurs through a Hopf bifurcation. In [15], in order to obtain the critical value of τ where the Hopf bifurcation occurs, we first, through asymptotic expansions, write our system as an PDE coupled with an

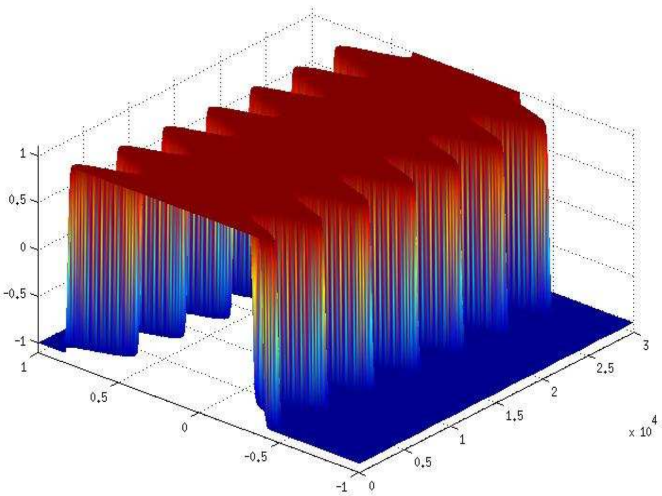

Figure 3: The mesa oscillates in time (red is $u = 1$ and blue is $u = -1$)

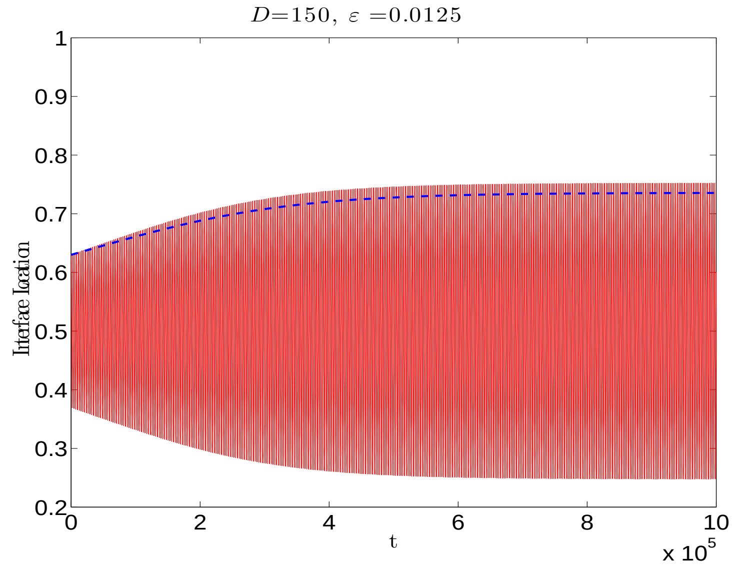

Figure 4: The location of the interface (in red) in time as well as the amplitude as predicted from asymptotics (in blue)

ODE, then as a system of ODEs. Then, using multiple scales analysis, we determine an equation for the amplitude of the oscillations from which we find that for $\tau > \tau_c$, the solution converges to a constant amplitude [15]. This method allows us to study the dynamics well beyond the Hopf bifurcation.

The numerical solution of (3), as seen in Figure 4, is performed using the spatial and temporal adaptive software package BACOL (see [17],[18]). Similar results were also obtained by using the method of lines and an ODE solver in Matlab.

Future work here consists of examining more than one mesa. A brief numerical study of this suggests that similar behaviour occurs as was determined for one mesa.

MSc research

"The Generalized Minimal Residual Method applied to the Inverse Problem of Electrocardiography"

Abstract: The inverse problem of electrocardiography—requiring the calculation of the heart surface potentials from the body surface potentials—presents a challenge because it is mathematically ill-posed. The currently accepted way of overcoming this is to use Tikhonov regularization. A key component of an effective regularization method is choosing a suitable value for the regularization parameter λ. Four commonly used methods (L-curve, CRESO, zero-crossing and discrepancy principle) as well as the new norm summation method are examined in this work. By calculating the optimal regularization parameter, using an a priori solution, the effectiveness of these methods is compared. An alternative way of solving the inverse problem is using a Krylov subspace method called the Generalized Minimal Residual (GMRES) method. This method forms an orthogonal basis of the sequence of successive matrix powers times the initial residual. The approximations to the solution are then obtained by minimizing the residual over the subspace formed. The GMRES method is able to solve the inverse problem without oversmoothing the solution, so as not to lose valuable localized electrical behaviour of the heart. The GMRES method and the Tikhonov regularization are compared and it is found that they perform similarly. Using both methods (one as a verification of the other) may help obtain more accurate assessments of the epicardial potentials.

The work here in presented in the preprint [16].

Current Research and Future Research

The focus of my research has been on techniques for examining formation of localized structures and especially the stability of these structures. There are many directions in which these techniques can be extended and applied.

Extension to two spatial dimensions

The same techniques as were used in the preprint [14] can be applied to the problem (1) but in two spatial dimensions. Here, as before, we can calculate the instability thresholds as well as the dynamics of the interfaces (which are now two dimensional).

Extension to a three-component system

Suzuki et al [19] considered a three-component system describing two inhibitors and one activator and showed that quasiperiodic motion of the interfaces was possible for certain parameters. By applying the techniques of [15] to a three-component system, analogous to (3), similar amplitude equations describing the motion of the interfaces may demonstrate spatio-temporal chaos.

Delay equations

In order to examine more complex behaviour of these mesa patterns, delay is introduced into one of the reaction terms of (3). Again, we consider if it is possible to produce spatio-temporal chaos. See [20, 21, 22].

Ecological/Biophysical applications

The analytical techniques in my previous work can be applied to systems that describe various ecological/biophysical phenomena ([12], [23], [26]) that exhibit solutions consisting of patterns as discussed. One possible avenue of research is to extend the previous techniques to models that involve non-local interaction, such as a model for animal group formation and movement ([23], [24], and [25]) or a model for cell aggregation and cancer invasion ([26]).

Numerical considerations

The use of numerical tools is critical to verify asymptotic results that come from analytical techniques used in my previous work. Since these systems consist of partial differential equations with small parameters and behaviour occuring at different time scales, solving numerically poses a number of difficulties. Adapting and expanding on current numerical techniques and tools is an essential part of work in this area.

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