MAT 1341B, INTRODUCTION TO LINEAR ALGEBRA, FALL 2003

Answers to Second Midterm, November 7, 2003

Prof. P. Selinger

Question 1. Suppose A is a 10×8-matrix (i.e., it has 10 rows and 8 columns). Further suppose that A has rank 3. Then how many parameters are there in the general solution of the homogeneous systems of equations $Ax = 0$?

Answer: *There are 8 variables, of which 3 pivot variables, which leaves 5 free variables, so the answer is 5.*

Question 2. If $B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & -1 \\ 0 & 2 & -1 \end{bmatrix}$, then what is the diagonal of B^{-1}?

Answer: *We invert B:*

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \iff \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\iff \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{array}\right] \iff \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{array}\right]$$

$$\iff \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & -2 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{array}\right]$$

So the diagonal of B^{-1} is $(3, -1, 1)$.

Question 3. Recall that two matrices are *row equivalent* if each can be obtained from the other by row operations. Which, if any, of the following three matrices are row equivalent?

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix},$$

Answer: *We reduce each of the matrices to row-canonical form. A is already in row-canonical form.*

$$B = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

So A and B are row equivalent to each other, but not to C.

Question 4. Which of the following statements are true in a vector space V of dimension n?

I. Any $n+1$ vectors in V are linearly dependent.

II. Any $n+1$ vectors in V form a spanning set of V.

III. Any $n-1$ vectors in V are linearly independent.

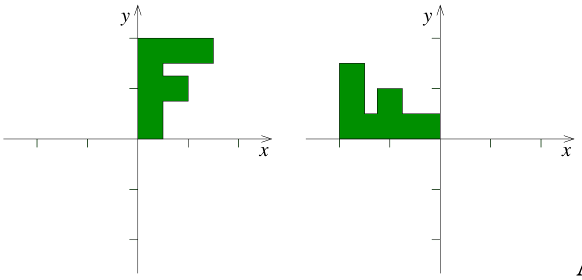
IV. Any n vectors in V form a basis.

V. Any basis of V consists of n vectors.

Answer: *I and V are true. II is false; for instance, take 4 vectors in $\mathbb{R}^3$ which lie in a plane. III is false; for instance, take 2 vectors in $\mathbb{R}^3$ which are collinear. IV is false; for instance, take 3 vectors in $\mathbb{R}^3$ which lie in a plane.*

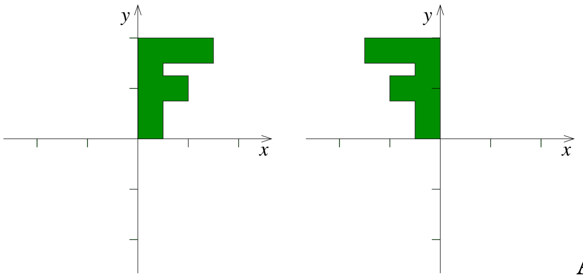
Question 5. For each of the following three linear functions $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$, give its matrix representation, with respect to the standard basis on $\mathbb{R}^2$.

(a) f is a rotation by 90 degrees, as shown in the illustration:



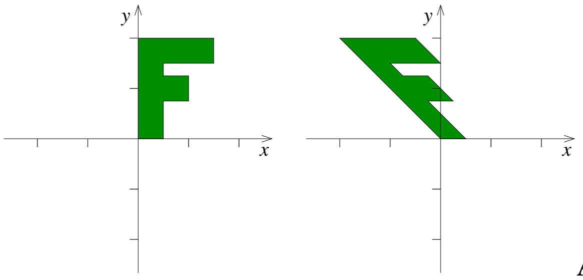
Answer: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(b) g is a reflection about the y-axis:



Answer: $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) h is the shearing shown in this illustration:



Answer: $C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Question 6. Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors $u_1 = (1, 2, -1, 3, 4)$, $u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$, $u_4 = (1, 4, 5, 1, 8)$, $u_5 = (2, 7, 3, 3, 9)$.

(a) Find a *subset* of the vectors $u_1, \dots, u_5$ which form a basis of W.

Answer: *We use the casting-out algorithm. This means, we write the vectors $u_1, \dots, u_5$ as the columns of a matrix and row-reduce.*

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are columns 1, 3, and 5. It follows that vectors u_1, u_3, and u_5 form a basis of W.

(b) What is the dimension of W?

Answer: *It is 3.*

(c) Find a basis of $\mathbb{R}^5$ which contains the vectors u_1 and u_3.

Answer: *We know that u_1 and u_3 are linearly independent. We need to add 3 linearly independent vectors to obtain a basis. Row-reduce:*

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 1 & 3 & 2 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -1 & 2 \end{bmatrix}$$

Since the pivot entries are in columns 1 and 2, we see that the vectors $(0, 0, 1, 0, 0)$, $(0, 0, 0, 1, 0)$, and $(0, 0, 0, 0, 1)$ are linearly independent of u_1 and u_3, thus the five vectors form a basis.

Question 7. Consider the following basis of $\mathbb{R}^3$:

$$S = \{(0, 0, 1), (0, 1, 2), (1, 2, 3)\}.$$

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$F(x, y, z) = (x - y, x - z, x + y - 2z).$$

(a) Let $v = (1, 1, 1)$. What are the coordinates of v with respect to the basis S?

Answer: *The coordinates are $[v]_S = [a, b, c]^T$, where $(1, 1, 1) = a(0, 0, 1) + b(0, 1, 2) + c(1, 2, 3)$. Solving the system of equations, we find $a = 0$, $b = -1$, $c = 1$, so $[v]_S = [0, -1, 1]^T$.*

(b) Let $w = (x, y, z)$. What are the coordinates of w with respect to the basis S?

Answer: *Let*

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Then P is the change-of-coordinates matrix from S to the standard basis. P^{-1} is the change-of-coordinates matrix from the standard basis to S. We calculate

$$P^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $[w]_S = P^{-1} \cdot [x, y, z]^T = (x - 2y + z, -2x + y, x)$.

(c) What is the matrix of F with respect to the standard basis $T = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$?

Answer: *The matrix of F w.r.t. the standard basis is*

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix}.$$

(d) What is the matrix of F with respect to the basis S?

Answer: *We apply F to the three basis vectors: $F(0, 0, 1) = (0, -1, -2)$, $F(0, 1, 2) = (-1, -2, -3)$, $F(1, 2, 3) = (-1, -2, -3)$. We express the results in coordinates relative to S: $[(0, -1, -2)]_S = [0, -1, 0]^T$, $[(-1, -2, -3)]_S = [0, 0, -1]^T$. Therefore the desired matrix is*

$$B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

One may also calculate it as $B = P^{-1}AP$.

Question 8. (a) Let $F : V \to W$ be a function between two vector spaces. Write down the axioms that F must satisfy in order to be linear.

Answer: *1. for all $v, u \in V$, $F(v + u) = F(v) + F(u)$. 2. for all $v \in V$, $k \in K$, $F(kv) = kF(v)$.*

(b) Prove that the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ which is given by $F(x, y, z) = (x, y + 2z)$ is linear.

Answer: *1. Let $v = (x, y, z)$, $u = (x', y', z')$. Then $F(v + u) = F(x + x', y + y', z + z') = (x + x', y + y' + 2(z + z')) = (x, y + 2z) + (x', y' + 2z') = F(x, y, z) + F(x', y', z') = F(v) + F(u)$. 2. Let $v = (x, y, z)$ and k a scalar, Then $F(kv) = F(kx, ky, kz) = (kx, ky + 2kz) = k(x, y + 2z) = kF(x, y, z) = kF(v)$.*

(b) Prove that the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ which is given by $F(x, y, z) = (x, y + 2)$ is *not* linear, by giving a *concrete* counterexamples to one of the axioms from (a).

Answer: *For example, it fails to satisfy axiom 1 when $v = (0, 0, 0)$ and $u = (0, 0, 0)$. In this case we have $F(v + u) = F(0, 0, 0) = (0, 2)$, whereas $F(v) + F(u) = F(0, 0, 0) + F(0, 0, 0) = (0, 2) + (0, 2) = (0, 4)$.*