F00 MAT 1341A Diagnostic Solutions

P_1 P_2

1. The planes 5x + 7y - 4z - 8 = 0 and x - y + 8 = 0 have a line in common. An equation for this line is (x, y, z) =

A normal for P_1 is $(5,7,-4)=n_1$

" " P_2 is $(1,-1,0)=n_2$

- A. (-4, 4, 0) + t (1, 1, 3)
- B. (4, 4, 0) + t (1, 1, 8)
- C. (0, 4, -4) + t (1, 3, -1)
- D. (4, 0, 4) + t (-1, 3, 1)
- E. (4, -4, 0) + t (-1, 1, -3)
- F. (-4, 4, 0) + t (1, 3, 1)

$\therefore$ the direction of the line is

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 5 & 7 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$(-4, -4, -12)$, or $(1, 1, 3)$. The only possibility is therefore A.

2. If u = (3, 3, 6) and v = (2, -1, 1), then the length of the projection of u along v is:

- A. $\frac{3\sqrt{6}}{2}$
- B. $\frac{\sqrt{6}}{2}$
- C. 0
- D. $\frac{3\sqrt{2}}{2}$
- E. $\frac{2\sqrt{2}}{3}$
- F. $\frac{2\sqrt{6}}{3}$

$$\|\text{proj}_v u\| = \left\| \frac{u.v}{\|v\|^2} v \right\|$$

$$= \frac{|u.v| \cancel{\|v\|}}{\|v\|^{\cancel{2}}} = \frac{|(3,3,6).(2,-1,1)|}{\sqrt{2^2+(-1)^2+(1)^2}}$$

$$= \frac{9}{\sqrt{6}} = \frac{9\sqrt{6}}{6} = \frac{3}{2}\sqrt{6}$$

3. Parametric equations of the line containing (-5, 0, 1) and which is parallel to the two planes $2x - 4y + z = 0$ and $x - 3y - 2z = 1$ are:

A. $x = -5 + 11t, \quad y = -3t, \quad z = 1 + 2t$
B. $x = -5 + 11t, \quad y = 5t, \quad z = 1 - 2t$
C. $x = -5t, \quad y = 0, \quad z = t$
D. $x = 5 + 11t, \quad y = 3t, \quad z = 1 + 2t$
E. $x = -5 + 5t, \quad y = -5t, \quad z = 1 - 10t$
F. $x = 5t, \quad y = 0, \quad z = t$

A direction vector for the line may be taken as the cross product of the 2 normals:

$$\begin{vmatrix} i & j & k \\ 2 & -4 & 1 \\ 1 & -3 & -2 \end{vmatrix} = (11, -(-5), -2) = (11, 5, -2).$$

Hence, the only possibility is B.

4. Find the point of intersection of the plane $2x + 2y - z = 5$ and the line with parametric equations $x = 4 - t, \quad y = 13 - 6t, \quad z = -7 + 4t$.

A. (-1, 2, 0)
B. (0, 1, 1)
C. (2, 1, 0)
D. (2, 1, 1)
E. (-1, -1, 2)
F. (-1, 2, 2)

Substituting into, we obtain

$$2(4-t) + 2(13-6t) - (-7+4t) = 5,$$

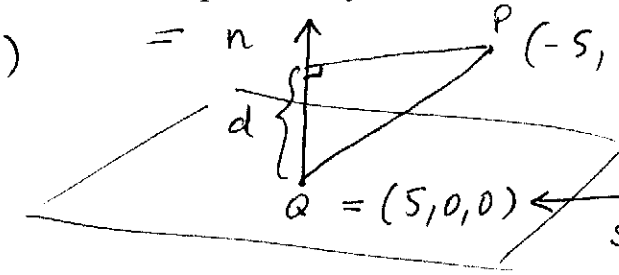
$$\Rightarrow t = 2.$$

$$\therefore (x, y, z) = (2, 1, 1).$$

5. How far is the point (-5, 0, 2) from the plane x - y = 5 ?

A. $5\sqrt{2}$
B. 10
C. 5
D. $10\sqrt{2}$
E. 2
F. $2\sqrt{5}$

$(1,-1,0) = n$



$$d = \left\| \operatorname{proj}_n \vec{PQ} \right\| = \frac{|\vec{PQ} \cdot n|}{\|n\|}$$

$$= \frac{|(-10, 0, 2) \cdot (1, -1, 0)|}{\sqrt{2}} = \frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

6. Find the distance from the point (5, 4, 7) to the line containing the points (3, -1, 2) and (3, 1, 1).

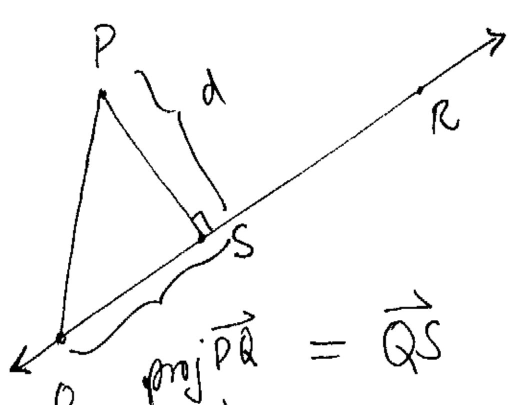
P Q R

$\vec{PQ} = (2, 5, 5)$ $\quad \vec{QR} = (0, -2, 1)$

A. 1
B. 11
C. 3
D. 9
E. 5
F. 7

$$d = \| \vec{QP} - \vec{QS} \|$$

$$= \| (-2, -5, -5) - \operatorname{proj}_{\vec{QR}} \vec{PQ} \|$$



$\operatorname{proj}_{\vec{QR}} \vec{PQ} = \vec{QS}$

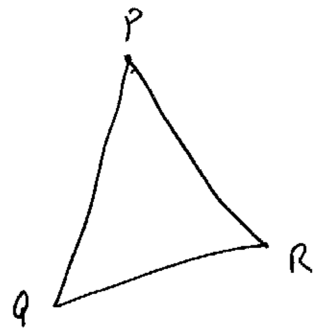
$$\operatorname{proj}_{\vec{QR}} \vec{PQ} = \frac{(\vec{PQ} \cdot \vec{QR})}{\|\vec{QR}\|^2} \vec{QR} = \frac{5}{5} \cdot (0, -2, 1)$$

$$\therefore d = \| (-2, -5, -5) - (0, -2, 1) \| = \| (-2, -3, -6) \|$$

$$= \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

7. Find the area of the triangle whose vertices are $\overset{P}{(1, 1, -1)}$, $\overset{Q}{(2, 0, 1)}$ and $\overset{R}{(1, -1, 3)}$.

A. 0
B. $\sqrt{5}$
C. 5
D. $4\sqrt{5}$
E. 10
F. $2\sqrt{5}$



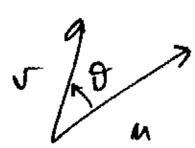
$$\text{area} = \frac{1}{2} \| \vec{QP} \times \vec{QR} \|$$

$$\begin{vmatrix} i & j & k \\ -1 & 1 & -2 \\ -1 & -1 & 2 \end{vmatrix} = (0, +4, 2)$$

& $\|(0,4,2)\| = 2\sqrt{5}$. $\therefore$ area $= \frac{2\sqrt{5}}{2} = \sqrt{5}$.

8. If $\|u\| = \|v\| = \sqrt{2}$ and $u \cdot v = 1$, then $\|u \times v\|^2 =$ $\|u\|^2 \|v\|^2 |\sin\theta|^2$

A. 1
B. 3
C. 0
D. $\sqrt{2}$
E. 2
F. 1/2



$$= \|u\|^2 \|v\|^2 (1 - \cos^2\theta)$$

$$= \|u\|^2 \|v\|^2 - \underbrace{(\cos^2\theta) \|u\|^2 \|v\|^2}_{(u \cdot v)^2}$$

$$= 2 \cdot 2 \cdot - 1$$

$$= 3$$

9. An equation for the plane passing through the points (2, 1, -1), (3, 2, 1) and which is parallel to the x-axis is:

A. $x + y + z = 2$
B. $x + y - z = 4$
C. $2x - z = 5$
D. $x - y = 1$
E. $2y - z = 3$
F. $-3x + 7y - 2z = 3$

Since the plane is ∥ to the x-axis, the coefficient of x in any equation for the plane is zero.* Hence E is the only possibility.

*[If $ax+by+cz=d$ is an eqn, we know $(1,0,0)\cdot(a,b,c) = 0 \Rightarrow a=0$]

10. What is the point of intersection of the lines

$x + 1 = 4t$ $\quad$ $x + 13 = 12\cancel{t}s$
$y - 3 = t$ $\quad$ and $\quad$ $y - 1 = 6\cancel{t}s$?
$z - 1 = 0$ $\quad$ $z - 2 = 3\cancel{t}s$

Change the parameters!

A. (13, -1, -2)
B. (-17, -1, 1)
C. (1, -3, -1)
D. (13, -5, 2)
E. (-2, 3, 1)
F. (4, -12, -1)

(We may assume that the lines intersect)

z : $1 = 3s + 2 \Rightarrow s = -\frac{1}{3}$
$\Rightarrow y = -1$

$\Rightarrow x = -17$

$\therefore$ B. is correct

11. Convert the complex number $\frac{(16 + 13i)(1 + 2i)}{10 + 5i}$ to the form a + i b.

A. $1 + 4i$
B. $-1/5 + (4/5)i$
C. 3
D. $4i$
E. $(1/5) + (4/5)i$
F. $1 - 4i$

$A = (16 + 13i)(1 + 2i) = (16 - 26) + i(45)$

$= -10 + 45i$

$\therefore \frac{A}{10 + 5i} = \frac{(-10 + 45i)(10 - 5i)}{125}$

$= \frac{(-2 + 9i)(2 - i)}{5} = \frac{5 + 20i}{5}$

$= 1 + 4i$

12. The polar form of the complex number $\frac{1 - \sqrt{3}i}{-1 + i}$ is:

$z = \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$

A. $\sqrt{2}(\cos(-5\pi/12) + i \sin(-5\pi/12))$
B. $\sqrt{2}(\cos(-\pi/12) + i \sin(-\pi/12))$
C. $\sqrt{2}(\cos(\pi/12) + i \sin(\pi/12))$
D. $\sqrt{2}(\cos(11\pi/12) + i \sin(11\pi/12))$
E. $\sqrt{2}(\cos(5\pi/12) + i \sin(5\pi/12))$
F. $\sqrt{2}(\cos(-7\pi/12) + i \sin(-7\pi/12))$

$= \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$

$|1 - \sqrt{3}i| = \sqrt{1^2 + \sqrt{3}^2} = 2. \quad \therefore \cos\theta_1 = \frac{1}{2}, \ \sin\theta_1 = -\frac{\sqrt{3}}{2} \Rightarrow \theta_1 = -\pi/3$

$|-1 + i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2} \quad \therefore \cos\theta_2 = \frac{-1}{\sqrt{2}}, \ \sin\theta_2 = \frac{1}{\sqrt{2}} \quad \therefore \theta_2 = 3\pi/4$

$\therefore \theta_1 - \theta_2 = -\pi/3 - 3\pi/4 = -\frac{\pi}{12} \cdot (4 + 9) \ ; \ \theta_1 - \theta_2 + 2\pi = \frac{11\pi}{12} \ ; \ \frac{r_1}{r_2} = \sqrt{2}$

$\therefore$ D. is correct