1. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 9$, find $\begin{vmatrix} 3a-5g & g & d \\ 3b-5h & h & e \\ 3c-5i & i & f \end{vmatrix} \overset{5C_2+C_1 \to C_1}{=} \begin{vmatrix} 3a & g & d \\ 3b & h & e \\ 3c & i & f \end{vmatrix}$

$= 3 \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$

$= 3 \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= -3 \cdot 9 = -27$

A. -45
B. 45
C. -27
D. 9
E. 27
F. -9

2. Which of the following are bases for R^3 ?

(1) { (4, 2, 0), (0, 1, 2), (1, 3, -1) }
(2) { (-1, 2, 3), (3, 3, 2) }
(3) { (-1, 3, -5), (1, -2, 4), (2, 0, 4), (5, 1, 9) }

$\begin{vmatrix} 4 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 2 & -4 \\ 0 & 1 & 0 \\ 1 & 3 & -7 \end{vmatrix}$

$= +1 \begin{vmatrix} 4 & -4 \\ 1 & -7 \end{vmatrix} = -24 \neq 0$

A. (1) and (2)
B. (1) only
C. (2) and (3)
D. None of them
E. All three
F. (2) only

3. Suppose A = $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{pmatrix}$. Which one of the following statements is true for A^{-1} ?

A. None of the below is true.
B. The second row is [1 2 -1].
C. The first row is [2 0 -1].
D. The third row is [-1 -1 1].
E. A^{-1} does not exist.
F. The second column is $[0 \ \ 2 \ \ -1]^t$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

4. Compute $\left(\begin{array}{cc|cc} -1 & 0 & -3 & 0 \\ 0 & -1 & 0 & -3 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)^{1001} = \begin{bmatrix} -I_2 & -3_2 \\ 0 & I_2 \end{bmatrix}^{1001}$

A. $\begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

B. $\begin{pmatrix} -1 & 0 & -3 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

C. $\begin{pmatrix} -1 & 0 & -3^{1001} & 0 \\ 0 & -1 & 0 & -3^{1001} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

D. $\begin{pmatrix} 1001 & 0 & -3003 & 0 \\ 0 & 1001 & 0 & -3003 \\ 0 & 0 & -1001 & 0 \\ 0 & 0 & 0 & -1001 \end{pmatrix}$

E. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

F. $\begin{pmatrix} -1001 & 0 & -3003 & 0 \\ 0 & -1001 & 0 & -3003 \\ 0 & 0 & 1001 & 0 \\ 0 & 0 & 0 & 1001 \end{pmatrix}$

$$\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \therefore \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}^{2n} = I$$

$$\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}^{3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}^{2n+1} = \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}$$

5. If $K = \{A \in M_{33} \mid A = -A^t\}$ is the subspace of anti-symmetric 3 by 3 matrices, then dim K is:

A. 3
B. 2
C. 6
D. 9
E. 4
F. 0

$$K = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

$$= \text{Span}\underbrace{\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}}_{\text{l.i.}}$$

6. Find all (a, b, c) so that $\begin{pmatrix} a & 1 & b & b & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & c \end{pmatrix}$ is in reduced row-echelon form.

must have $c = 0$

Must have $b = 0$

A. (0, 0, 1)
B. (1, 0, 0)
C. (1, 1, 1)
D. (0, 0, 0)
E. (1, 0, 0) and (0, 0, 1)
F. (1, 0, 0) and (0, 0, 0)

a can be 0 or 1.

$\therefore$ (1, 0, 0), (0, 0, 0)

7. Consider the linear system

T

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 0 \\ -x & + & y & - & z & = & 0 \\ x & + & ky & + & z & = & 0 \end{array}$$

a) If $[A\,|\,0]$ is the augmented matrix of the system above, find rank A and rank$[A\,|\,0]$ for all values of k.

b) Find all k so that this system has
  i) a unique solution,
  ii) infinitely many solutions, and
  iii) no solutions.

c) In case (ii) above, give a complete geometric description of the set of solutions.

a) $$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & k & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k+1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & k+1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]. \ \left(\tfrac{1}{2}\right)$$

$\therefore$ rank $A = \begin{cases} 2 & \text{if } k=-1 \ \left(\tfrac{1}{2}\right) \\ 3 & \text{if } k \neq -1 \ \left(\tfrac{1}{2}\right) \end{cases}$ & rank$[A|0]$ = rank A in all cases. $\left(\tfrac{1}{2}\right)$

(system is always consistent since rank A = rank $[A|0]$)

b) i) unique soln $\Leftrightarrow$ rank $A = 3$ $\Leftrightarrow$ $k \neq -1$ $\left(\tfrac{1}{2}\right)$ $\left(\tfrac{1}{2}\right)$ – reasons

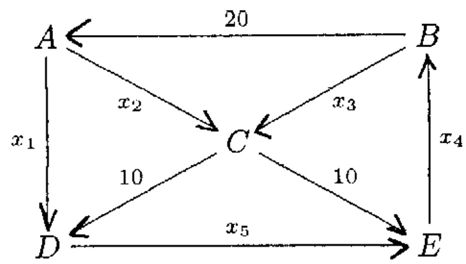
ii) ∞ly many soln $\Leftrightarrow$ rank $A < 3$ $\Leftrightarrow$ $k = -1$ $\left(\tfrac{1}{2}\right)$

iii) this never occurs (system is homogeneous) $\left(\tfrac{1}{2}\right)$

c) $k=-1$: $$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} & s & & \end{array} \\ \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{array}{l} x = s \ \left(\tfrac{1}{2}\right) \\ y = s \\ z = 0 \end{array} \ s \in \mathbb{R},$$

$\therefore$ this is the line through O with direction $(1,1,0)$. $\left(1\tfrac{1}{2}\right)$

8. Consider the closed network of streets and intersections below. The arrows indicate the
8 direction of traffic flow along the one–way streets, and the numbers are the flows observed during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



2 a) Using Kirchoff's law, write down the linear system which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b)!)

1 b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{c} \begin{array}{ccccc} & & & & s \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints at this point.)

2 c) Using (b) and the constraints from (a), find all possible traffic flows. (You do not need to list them all individually: simply give the possible vales of parameter(s).). How many different network flows are there?

1 d) Are there any network flows with both AD and BC closed for roadwork? What are the possible flows on the other streets in this case?

a)

	Flow in	=	Flow out
A	20	=	$x_1 + x_2$
B	x_4	=	$20 + x_3$
C	$x_2 + x_3$	=	20
D	$x_1 + 10$	=	x_5
E	$10 + x_5$	=	x_4

(1)

($\frac{1}{2}$) $x_i \geq 0$ and $x_i \in \mathbb{Z}$ ($\frac{1}{2}$) $i = 1, \ldots, 5$

b) $(x_1, x_2, x_3, x_4, x_5) = (-10+s,\ 30-s,\ -10+s,\ 10+s,\ s)$ $s \in \mathbb{R}$ (

c) $x_5 \geq 0 \Leftrightarrow s \geq 0$, $x_1 = x_3 \geq 0 \Leftrightarrow s \geq 10$; $x_2 \geq 0 \Leftrightarrow 30 \geq s$; $x_4 \geq 0 \Leftrightarrow s \geq -10$; $x_i \in \mathbb{Z} \Leftrightarrow s \in \mathbb{Z}$. ($\frac{1}{2}$) -reasons ($\frac{1}{2}$) ($\frac{1}{2}$) Hence the possible values of s are $10, 11, \ldots, 30$. (integer v...)
There are 21 ($\frac{1}{2}$) different flows. d) yes (when $s = 10$) ($\frac{1}{2}$). Then $x_2 = 20$, $x_4 = 20$, $x_5 = 10$. ($\frac{1}{2}$)

89. Let $W = \text{span}\{(-1,1,0),(1,1,-2)\}$ and $L = \text{span}\{(1,1,1)\}$.

For $v \in \mathbf{R}^3$, let $\text{proj}_W(v)$ and $\text{proj}_L(v)$ denote the orthogonal projections of v onto W and L respectively.

a) Give complete geometric descriptions of W and L.

b) Find a basis of $W^\perp = \{v \in \mathbf{R}^3 \mid v \cdot w = 0 \text{ for all } w \in W\}$ and use this to show that $L = W^\perp$.

c) If $(x,y,z) \in \mathbf{R}^3$, find a formula for $\text{proj}_W(x,y,z)$.

d) If $(x,y,z) \in \mathbf{R}^3$, show that $\text{proj}_L(x,y,z) = (\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3})$. *

e) Use (c) and (d) to show that $v = \text{proj}_W(v) + \text{proj}_L(v)$ for all $v \in \mathbf{R}^3$.

2 a) W is the plane 1 through 0 with normal $\begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{vmatrix} = (-2,-2,-2)$.
L is the line 1 through 0 with direction $(1,1,1)$.

$1\frac{1}{2}$ b) $W^\perp = \left\{(x,y,z) \,\middle|\, \begin{matrix} -x+y=0 \\ x+y-2z=0 \end{matrix}\right\}$ just! $\frac{1}{2}$ $\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$ $\therefore$ $W^\perp = \{(s,s,s) \mid s \in \mathbb{R}\} = \text{span}\{(1,1,1)\}$. Since $(1,1,1) \neq 0$, $\{(1,1,1)\}$ is a basis of $W^\perp$. 1 Since $L = \text{span}\{(1,1,1)\}$, $L = W^\perp$.

1 c) $\{\underset{v_1}{(-1,1,0)}, \underset{v_2}{(1,1,-2)}\}$ is an orthogonal basis for W, so

$\text{proj}_W(x,y,z) = \frac{(x,y,z)\cdot v_1}{\|v_1\|^2} + \frac{(x,y,z)\cdot v_2}{\|v_2\|^2}$ 1 $= \frac{-x+y}{2}(-1,1,0) + \frac{(x+y-2z)}{6}v$

$= \frac{1}{6}\{(3x-3y, -3x+3y, 0) + (x+y-2z,\ x+y-2z,\ -2x-2y+4z)\}$

$= \frac{1}{6}(4x-2y-2z,\ -2x+4y-2z,\ -2x-2y+4z)$

$\frac{1}{2}$ d) $\text{proj}_L(x,y,z) = \frac{(x,y,z)\cdot(1,1,1)}{3}(1,1,1) = \frac{x+y+z}{3}(1,1,1) =$ above*. $\frac{1}{2}$

e) x component: $\frac{1}{6}(4x-2y-2z) + \frac{2(x+y+z)}{6} = \frac{6x}{6} = x$

1 y " $\frac{1}{6}(-2x+4y-2z) +$ " $= 6y/6 = y$

z " $\frac{1}{6}(-2x-2y+4z) +$ " $= 6z/6 = z$

$\therefore \text{proj}_W v + \text{proj}_L v = v$ for all v.

T **10.** Let $A = \frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ and $v_1 = (1,1,1)$ and $v_2 = (1,-1,0)$ be two vectors in $\mathbf{R}^3$.

(1) a) Show that $Av_1 = v_1$ and $Av_2 = v_2$, and hence show that $\{v_1, v_2\} \subseteq \mathrm{col}\,(A)$, where $\mathrm{col}\,(A)$ denotes the column space of P.

($1\frac{1}{2}$) b) Show that rank $A = 2$, and use this to deduce that $\dim \mathrm{col}\,(A) < 3$.

(2) c) Use (a) and (b) to show that $\{v_1, v_2\}$ is a basis of $\mathrm{col}\,(A)$, and give a complete geometric description of $\mathrm{col}\,(A)$.

(1) d) Show that $\ker A = \mathrm{span}\{v_1 \times v_2\}$.

a) $Av_1 = \frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}6\\6\\6\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} = v_1$; $Av_2 = \frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix} = \frac{1}{6}\begin{bmatrix}6\\-6\\0\end{bmatrix} = v_2$.

($\frac{1}{2}$) ($\frac{1}{2}$)

Since $A = [C_1\ C_2\ C_3]$ (C_i = ith col of A), $Av_1 = [C_1\ C_2\ C_3]\begin{bmatrix}1\\1\\1\end{bmatrix} = C_1 + C_2 + C_3 =$ so $v_1 \in \mathrm{span}\{C_1, C_2, C_3\}$. Similarly, $Av_2 = C_1 - C_2 = v_2$, so $v_2 \in \mathrm{span}\{C_1, C_2$

($\frac{1}{2}$)

b) $\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 2 \\ 0 & 24 & 12 \\ 0 & 12 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ ($\frac{1}{2}$) $\therefore$ rank $A = 2$. If $\dim \mathrm{col}(A) =$ 3 the cols of A would be l.o.i., which would mean that A is invertible, whence rank $A = 3$. But rank $A < 3$, so $\dim \mathrm{col}(A) < 3$. (1)

c) Since $\{v_1, v_2\}$ is l.o.i., $\dim \mathrm{col}(A) \geq 2$. By (b) $\dim \mathrm{col}(A) \leq 2$, so $\dim \mathrm{col}(A) = 2$. Hence $\{v_1, v_2\}$ is a basis of $\mathrm{col}(A)$. (1) The column space is the plane through O with normal $v_1 \times v_2$ (1)

$= \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1, 1, -2)$.

d) $\ker A = \ker \begin{bmatrix} 1 & -5 & -2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = \{(-\frac{1}{2}s, -\frac{1}{2}s, s) \mid s \in \mathbb{R}\}$ ($\frac{1}{2}$)

$= \mathrm{span}\{(-\frac{1}{2}, -\frac{1}{2}, 1)\}$

$= \mathrm{span}\{(1, 1, -2)\}$ ($\frac{1}{2}$)

$= \mathrm{span}\{v_1 \times v_2\}$

A11. Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$.

a) Show that the eigenvalues of A are 0 and 3.

b) Find a basis B_0 of $E_0 = \{x \in \mathbf{R}^3 \mid Ax = 0\}$.

c) Find a basis of $E_3 = \{x \in \mathbf{R}^3 \mid (A - 3I)x = 0\}$.

d) Show that the set consisting of all vectors from the bases for E_0 and E_3 is a basis for $\mathbf{R}^3$.

e) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

a) $C_A(x) = \det(A - xI) = \begin{vmatrix} 2-x & -1 & -1 \\ -1 & 2-x & -1 \\ -1 & -1 & 2-x \end{vmatrix} \overset{-R_1+R_3 \to R_3}{=} \begin{vmatrix} 2-x & -1 & -1 \\ -1 & 2-x & -1 \\ -3+x & 0 & 3-x \end{vmatrix}$

$= (x-3)\begin{vmatrix} 2-x & -1 & -1 \\ -1 & 2-x & -1 \\ 1 & 0 & -1 \end{vmatrix} = (x-3)\begin{vmatrix} 2-x & -1 & 1-x \\ -1 & 2-x & -2 \\ 1 & 0 & 0 \end{vmatrix} = (x-3)\begin{vmatrix} -1 & 1-x \\ 2-x & -2 \end{vmatrix}$. (1)

$= (x-3)\left[\underbrace{2 - x^2 + 3x - 2}\right] = -(x-3)^2 x$. Hence $x = 3$ and $x = 0$ are evals.

b) $E_0 = \ker(A - 0I) = \ker\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \ker\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \{(s, s, s) \mid s \in \mathbb{R}\} = \operatorname{span}\{(1,1,1)\}$. Hence $\{\overset{v_1}{(1,1,1)}\}$ (1)

is a basis of E_0.

c) $E_3 = \ker(A - 3I) = \ker\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \ker\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{(-s-t, s, t) \mid s \in \mathbb{R}\}$

$= \operatorname{span}\{(-1,1,0), (-1,0,1)\}$. Hence $\{\overset{v_2}{(-1,1,0)}, \overset{v_3}{(-1,0,1)}\}$ is a basis of E_3. ($1\frac{1}{2}$)

d) $\det\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 1 & -1 & -2 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} = 3 \neq 0 \therefore \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. (1)

e). Set $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. ($\frac{1}{2}$) Since $\det P \neq 0$, P exists, ($\frac{1}{2}$) and we know

$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$ ($\frac{1}{2}$)

6 **12.** a) Let A be a real $n \times n$ matrix. Give 3 different statements which are equivalent to:

The columns of A are linearly dependent.

b) State whether the following are always true or could be false. If true, explain why, if false, give a numerical example to illustrate.

(i) Suppose an $n \times n$ matrix A satisfies $A^2 = A$. If v is an eigenvector of A with eigenvalue λ, then $\lambda = 0$ or $\lambda = 1$.

(ii) If an $n \times n$ matrix A satisfies $A^2 = 0$, then $A = 0$.

(iii) Let $\mathbf{F}[0,1] = \{f \mid f : [0,1] \to \mathbf{R}\}$ be the vector space of real-valued funtions defined on $[0,1]$. If f, g and h are any three different functions in $\mathbf{F}[0,1]$, then $\{f, g, h\}$ must be linearly independent.

a) See your notes. $(1) + (1) + (1)$

$(\frac{1}{2}) + (\frac{1}{2})$

b) (i) Suppose $Av = \lambda v$. Then $A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$

But $A^2 v = Av = \lambda v$, so $\lambda v = \lambda^2 v$ or $(\lambda - \lambda^2)v = 0$. Since $v \neq 0$,

$\lambda - \lambda^2 = 0$ so $\lambda = 0$ or $\lambda = 1$. This is always true.

(ii) This is false. e.g. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$(\frac{1}{2}) + (\frac{1}{2})$

but $A \neq 0$.

(iii) This is false. e.g. let $f = x$, $g = 2x$, $h = 0$.

Then $\{f, g, h\}$ is l.d. since $0f + 0g + 1.h = 0$.

$(\frac{1}{2}) + (\frac{1}{2})$