1. The vector $v = (6, 3, 5, 7, 7)$ can be written as $v = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$, where $\{v_1, v_2, v_3, v_4\}$ is the orthogonal set with

$v_1 = (1, 0, 1, 0, 1)$, $v_2 = (0, 1, -1, 1, 1)$, $v_3 = (-1, 0, 1, 1, 0)$, and $v_4 = (1, 0, 0, 1, -1)$,

Then, $(c_1, c_2, c_3, c_4) =$

A. (2, 3, 2, 2)
B. (6, 3, 2, 2)
C. (6, 2, 3, 3)
D. (2, 3, 3, 2)
E. (2, 6, 3, 2)
F. (6, 6, 2, 2)

$$c_1 = \frac{v \cdot v_1}{\|v_1\|^2} = \frac{18}{3} = 6$$

$$c_2 = \frac{v \cdot v_2}{\|v_2\|^2} = \frac{12}{4} = 3$$

$\vdots$

Hence the only possibility is B.

Check: $c_3 = \dfrac{v \cdot v_3}{\|v_3\|^2} = \dfrac{6}{3} = 2$

$$c_4 = \frac{v \cdot v_4}{\|v_4\|^2} = \frac{6}{3} = 2.$$

2. Which one of the following is a basis for the subspace

$W =$ $\{ (x, y, z) \mid x - y + z = 0 \}$ of R^3 ?

$x = y - z$

A. $\{(3, 0, -3)\}$
B. $\{(1, 0, 0), (1, 2, 1)\}$
C. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
D. $\{(1, 2, 1)\}$
E. $\{(1, 1, 0), (-1, 0, 1)\}$
F. $\{(-3, 0, 3), (1, 0, 0)\}$

Since W is a plane through 0, $\dim W = 2$. This excludes all but B, E & F. However, both B & F include the vector $(1,0,0) \notin W$. Hence E is correct

Directly: $W = \{ (y-z, y, z) \mid y, z \in \mathbb{R} \}$

$= \operatorname{span} \{ (1,1,0), (-1, 0, 1) \}$.

Moreover, $\{(1,1,0), (-1,0,1)\}$ is l.i. since the 2 vectors aren't parallel. This is then a basis of W.

4. Let $\mathbf{F}([-1,1]) = \{f \mid f : [-1,1] \to \mathbf{R}\}$ be the vector space of real-valued functions defined on $[-1,1]$. Recall that the zero of $\mathbf{F}[-1,1]$ is the function that has the value 0 for all $x \in [-1,1]$.

Define three functions in $\mathbf{F}([-1,1])$ by $f(x) = 1+x$, $g(x) = x+x^2$ et $h(x) = x+x^2+x^3$ and let $W = \text{span}\{f, g, h\}$.

a) Show that f, g and h are linearly independent.

b) Find a basis for W and the dimension of W.

c) If $j(x) = 1 - x^2 + x^3$ show that $j \in W$.

d) What is $\dim \text{span}\{f, g, h, j\}$?

a) Suppose $af + bg + ch = 0$ where $a, b, c \in \mathbb{R}$. i.e.

2 $\quad a(1+x) + b(x+x^2) + c(x+x^2+x^3) = 0$ (← zero function)

At $x = -1$, this yields $\quad -c = 0$

$x = 0 \quad a = 0$

$x = 1 \quad 2a + 2b + 3c = 0.$

$\Rightarrow a = c = 0$ and $b = 0$.

Hence $a = b = c = 0$, so $\{f, g, h\}$ is l.i.

b) Since $\{f, g, h\}$ spans W (by def^n of W!), and we

2 showed in (a) that $\{f, g, h\}$ is l.i., it is a basis of W. Hence $\dim W = 3$.

c) We solve $1 - x^2 + x^3 = af + bg + ch$*, where $a, b, c \in \mathbb{R}$.

1 By inspection, $x^3 = h - g$ and $1 - x^2 = f - g$, so $1 - x^2 + x^3 = f - 2g + h$. (Or, expand * and equate coefficients of x, and solve for a, b, c.)

d) Since $j \in \text{span}\{f, g, h\} = W$, $\text{span}\{f, g, h, j\} = W$.

1 Hence $\dim \text{span}\{f, g, h, j\} = \dim W = 3$.

5. Supoose $v_1 = (1,2,1)$ and $v_2 = (-1,1,-1)$ and let $W = \text{span}\{v_1, v_2\}$.

a) Show that $\{v_1, v_2\}$ is an orthogonal set. Is $\{v_1, v_2\}$ linearly independent? (Give reasons)

b) Give a complete geometric description of W.

c) Find an orthonormal basis B of W, and hence find $\dim W$.

d) Extend the basis B to an orthogonal basis of $\mathbf{R}^3$.

a) $v_1 \cdot v_2 = -1 + 2 - 1 = 0$; (1) moreover, $v_1 \neq 0$ & $v_2 \neq 0$. So, $\{v_1, v_2\}$ is orthogonal. Yes, $\{v_1, v_2\}$ is l.i., since any orthogonal set is l.i. (1/2)

b) Since $v_1 \not\parallel v_2$, W is the plane ($\frac{1}{2}$) through O ($\frac{1}{2}$) with normal vector $v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ -1 & 1 & -1 \end{vmatrix} = (-3, 0, 3)$ ($\frac{1}{2}$)

c) Since $\{v_1, v_2\}$ is l.i., and spans W (by def^n of W) ($\frac{1}{2}$) it is a basis of W. Moreover, it is orthogonal, by (a). So we only need to normalize: let $w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{6}}(1,2,1) = \frac{\sqrt{6}}{6}(1,2,1)$

and $w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{3}}(-1,1,-1)$ (1) $= \frac{\sqrt{3}}{3}(-1,1,-1)$ Then, $\{w_1, w_2\}$ is an orthonormal basis for W. So $\dim W = 2$. ($\frac{1}{2}$)

d) By (b), $v_3 = (-3,0,3) \neq 0$ is orthogonal to both v_1 & v_2, ($\frac{1}{2}$) so $\{v_1, v_2, v_3\}$ is orthogonal, and so l.i. Since $\dim \mathbb{R}^3 = 3$, this set of 3 vectors is an (orthogonal) basis of $\mathbb{R}^3$ ($\frac{1}{2}$)

6. Suppose $\{u, v\}$ is a linearly independent set in vector space V, and suppose $w \in V$ is such that $\{u, v, w\}$ is linearly **de**pendent.

a) State carefully what "$\{u, v\}$ is linearly independent" means. (i.e. give the definition.)

In the next two parts, either show that the statement is always true, or give a counterexample (in $\mathbf{R}^2$ or $\mathbf{R}^3$) to show it isn't always true.

b) $w \in \text{span}\{u, v\}$.

c) $u \in \text{span}\{v, w\}$.

1 – [illegible]

(where)

$2\frac{1}{2}$

a) $\{u, v\}$ is l.i. means that $au + bv = 0$ for $a, b \in \mathbb{R} \Leftrightarrow a = b = 0$.

b) This is always true ($\frac{1}{2}$) since we have a theorem that states

(2) that if $\{u, v\}$ is l.i., then $w \notin \text{span}\{u, v\} \Leftrightarrow \{u, v, w\}$ is l.i.

Since $\{u, v\}$ is l.i., and $\{u, v, w\}$ isn't l.i., we must have $w \in \text{span}\{u, v\}$.

$\frac{1}{2}$ without "theorem"

(OR: Since $\{u, v, w\}$ is l.d., $au + bv + cw = 0$ with $(a, b, c) \neq (0, 0, 0)$. If $c = 0$, then $au + bv = 0$ with $(a, b) \neq (0, 0)$, which contradicts the l.i. of $\{u, v\}$. So $c \neq 0$. Thus, $w = -\left(\frac{a}{c}\right)u - \left(\frac{b}{c}\right)v \in \text{span}\{u, v\}$)

c) This isn't always true. e.g. let $V = \mathbb{R}^2$, $u = (1, 0)$ and

($1\frac{1}{2}$) $v = w = (0, 1)$. Then $\{u, v\}$ is l.i., $\{u, v, w\}$ is l.d., but $u = (1, 0) \notin \text{span}\{v, w\} = \text{span}\{(0, 1)\}$.

[illegible]

excellent example.