Mini-Test 1 F.00 Solutions.

1. Which of the following are subspaces of $\mathbf{R}^3$?

$U = \{ (x, y+1, x-2y) \mid x, y \in \mathbf{R} \}$, $\not\ni (0,0,0)$
$V = \{ (x, 2y, x+y) \mid x, y \in \mathbf{R} \}$ ✓
$W = \{ (x, y, 3xy) \mid x, y \in \mathbf{R} \}$ $u=(1,0,0) \in W$ and $v=(0,1,0) \in W$ but $u+v = (1,1,0) \notin W$.

A. V only
B. U and W only
C. U and V only
D. U only
E. V and W only
F. W only

(V is the plane with equation $x + \frac{1}{2}y - z = 0$, which is a plane through the origin.
For later: $V = \text{span}\{(1,0,1), (0,2,1)\}$, and so is a subspace.)

2. Which of the following are vector spaces?

(1) $V = \{ (x, y, z) \in \mathbf{R}^3 \mid x - 2y + 3z = 0 \}$, with the usual operations. V is a plane through the origin in $\mathbb{R}^3$ ✓

(2) $W = \{ (x, y) \in \mathbf{R}^2 \mid xy = 1 \}$, with the usual operations. $(0,0) \notin W$.

(3) $X = \left\{ \begin{bmatrix} 0 & a \\ b & c \end{bmatrix} \middle| \; a, b \text{ and } c \in \mathbf{R} \right\}$, with the usual operations of M_{22}. ✓

A. (3) only
B. (1) only
C. (2) and (3)
D. (1) and (3)
E. (1) and (2)
F. (2) only

3. Which one of the following is a subspace of $\mathbf{R}^4$?

A. $\{ (a, b, c, d) \mid a + b + c + d = 1 \}$ X doesn't contain (0, 0, 0)
B. $\{ (a, b, c, d) \mid a > 0 \text{ and } b > 0 \}$ X ″
C. None of the others is a subspace of $\mathbf{R}^4$. ✓
D. $\{ (a, b, c, d) \mid a + b = 0, cd = 0 \}$ X contains (0,0,1,0) and (0,0,0,1), but not their sum.
E. $\{ (a, b, c, d) \mid a > 0 \text{ and } b < 0 \}$ X doesn't contain (0,0,0,0).
F. $\{ (a, b, c, d) \mid a = 1, b = 0 \text{ and } c + d = 1 \}$ X doesn't contain (0,0,0,0).

4. Which of the following are subspaces of $\mathbf{F}[0,1]$?

$V = \{f \in \mathbf{F}[0,1] \mid f(1) = 2 f(0)\}$ ✓ (try the subspace test)
$U = \{f \in \mathbf{F}[0,1] \mid f(1) = 0\}$ ✓
$W = \{f \in \mathbf{F}[0,1] \mid f(0) = 1 - f(1)\}$ doesn't contain the zero function

A. V and W.
B. W only.
C. U only .
D. None is a subspace of F[0,1] .
E. U and W.
F. U and V.

5. The volume of the parallelepiped with a vertex at the origin and edges given by the vectors $\underbrace{\mathbf{i}+\mathbf{j}+2\mathbf{k}}_{u}$, $\underbrace{2\mathbf{j}+5\mathbf{k}}_{v}$ and $\underbrace{\mathbf{i} \overset{+2j}{+} 6\mathbf{k}}_{w}$ is:

A. 14
B. 11
C. 7
D. 9
E. 3
F. 10

$$u \cdot v \times w = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 5 \\ 1 & 2 & 6 \end{vmatrix}$$

$$= 1 \times (2\times 6 - 2\times 5) - 1\,(0\times 6 - 1\times 5) + 2\,(0\times 2 - 1\cdot 2)$$

$$= 2 \quad + 5 \quad - 4$$

$$= 3$$

$\therefore$ vol $= |u \cdot v \times w| = |3| = 3.$

6. Consider the vector space $\mathbf{F}(\mathbf{R}) = \{f \mid f\colon \mathbf{R} \to \mathbf{R}\}$, with the standard operations. Recall that the zero of $\mathbf{F}(\mathbf{R})$ is the function that has the value 0 for all $x \in \mathbf{R}$.

Let $W = \{f \in \mathbf{F}(\mathbf{R}) \mid f(0) = f(\pi)\}$ be the subset of functions which have the same value at $x = 0$ and $x = \pi$.

a) Show that W is a subspace of $\mathbf{F}(\mathbf{R})$.

b) Show that $\sin x$ and $\sin^2 x$ belong to W.

c) Show that $\cos^2 x \in \operatorname{span}\{1, \sin^2 x\}$.

d) Show that $\cos^2 x \in W$ in two different ways. (*Hint: Use (c)!*)

(You must justify your answers.)

a) Since $O(0) = 0 = O(\pi)$, the zero function belongs to W. $\frac{1}{2}$
Suppose $f, g \in W$. Then $(f+g)(0) = f(0) + g(0) = f(\pi)+g(\pi) = (f+g)(\pi)$, so $f+g \in W$. (1)
If $s \in \mathbb{R}$ and $f \in W$, then $(sf)(0) = s.f(0) = s.f(\pi) = (sf)(\pi)$, so $sf \in W$. (1) Hence, by the subspace test, W is a subspace of $F(R)$.

b) Since $\sin(0) = 0 = \sin(\pi)$, $\sin x \in W$. Similarly, (1)
$\sin^2(0) = 0^2 = 0 = \sin^2(\pi)$, $\sin^2 x \in W$.

c) From the identity $\sin^2 x + \cos^2 x = 1$ (true for all x), we find $\cos^2 x = 1 - \sin^2 x$, (1) so $\cos^2 x \in \operatorname{span}\{1, \sin^2 x\}$.

d) 1st method: $\cos^2(0) = 1^2 = (-1)^2 = \cos^2(\pi)$, so $\cos^2 x \in W$ $\frac{1}{2}$

(1) 2nd method: $\cos^2 x$ is a linear combination of 1 and $\sin^2 x$. We know that $\sin^2 x \in W$; similarly, $1 \in W$ since $1(0) = 1 = 1(\pi)$. Because W is a subspace, and both $1, \sin^2 x \in W$, W contains any l.c. of 1 and $\sin^2 x$. In particular, $W \ni \cos^2 x$.

3rd method: We know that $1, \sin^2 x \in W$, so $\operatorname{span}\{1, \sin^2 x\} \subseteq W$ (like 2nd) by Thm 2, P205, part (b). But $\cos^2 x \in \operatorname{span}\{1, \sin^2 x\}$, so $\cos^2 x \in W$.

7. Let $v_0 = (1,0,1) \in \mathbf{R}$ and $U = \{(x,y,z) \in \mathbf{R}^3 \mid x+z=0\}$.

a) Show that $u \in U$ if and only if u is orthogonal to v_0.

b) If $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$, show that

$$u_1 = e_1 - \mathrm{proj}_{v_0}(e_1) \quad \text{and} \quad u_2 = e_2 - \mathrm{proj}_{v_0}(e_2)$$

are orthogonal, and that they both belong to U.

c) Show that $au_1 + bu_2 \in U$ for all real scalars a and b.

d) Conversely, show that if $u \in U$, then there are real scalars a and b such that

$$u = au_1 + bu_2.$$

e) Give a complete geometric description of U.

(You must justify your answers.)

a) $u = (x,y,z) \in U \Leftrightarrow x+z=0 \Leftrightarrow (x,y,z)\cdot(1,0,1) = 0$
$\Leftrightarrow u \cdot v_0 = 0 \Leftrightarrow$ u and v_0 are orthogonal. $\frac{1}{2}$

b) $u_1 = (1,0,0) - \dfrac{e_1 \cdot v_0}{2}(1,0,1) = \left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ $\frac{1}{2}$

$u_2 = (0,1,0) - \dfrac{e_2 \cdot v_0}{2}(1,0,1) = (0,1,0)$ $\frac{1}{2}$

Then $u_1 \cdot u_2 = \frac{1}{2} \times 0 + 0 \times 1 + \left(-\frac{1}{2}\right) \times 0 = 0$, so u_1 and u_2 are orthogonal $\frac{1}{2}$ Moreover, $u_1 \cdot v_0 = \frac{1}{2} - \frac{1}{2} = 0$ and $u_2 \cdot v_0 = 0$, $\frac{1}{2}$
so by (a), u_1 and u_2 belong to U.

c) Since $(au_1 + bu_2)\cdot v_0 = a\, u_1 \cdot v_0 + b\, u_2 \cdot v_0 = 0 + 0 = 0$, 1
$au_1 + bu_2 \in U$ for all a, b.

d) $U = \{(x,y,z) \mid x+z=0\} = \{(x,y,-x) \mid x,y \in \mathbb{R}\}$, so if u belongs to U, $u = (x, y - x) = x(1,0,-1) + y(0,1,0)$ 1
$= 2x \cdot \left(\frac{1}{2}, 0, -\frac{1}{2}\right) + y\, u_2 = 2x \cdot u_1 + y \cdot u_2.$

e) U is the plane in $\mathbb{R}^3$ passing through the origin with normal $(1,0,1)$. $1\frac{1}{2}$