1341A.F00 Mt2 Solutions

1. Find all values of c so that {(2, -1, 3), (0, c, 2), (8, -1, 8)} is linearly independent.

A. $c \neq \pm 3/2$
B. $c = 0$
C. $c = 3/2$
D. $c \neq -3/2$
E. $c > 0$
F. $c < 0$

$$A = \begin{bmatrix} 2 & 0 & 8 \\ -1 & c & -1 \\ 3 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & c & 3 \\ 0 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & c & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 3+2c \end{bmatrix},$$ so rank $A = 3 \iff$

$3+2c \neq 0$ ie

$c \neq -\frac{3}{2}$

2. A is a 10 by 6 matrix such that Ax = 0 has only the trivial solution. Answer the following questions:

- What is the rank of A?
- Is Ax = b consistent for all $b \in \mathbf{R}^{10}$?

A. 0, Yes
B. 6, Yes
C. 6, No
D. 4, Yes
E. 10, No
F. 10, Yes

Since $Ax = 0 \iff x = 0$, rank A = # cols of $A = 6$.

Since Ax is a l.c. of the cols. of A, $\{Ax \mid x \in \mathbb{R}^6\}$ is spanned by 6 vectors (the columns of A). Hence we cannot have $\{Ax \mid x \in \mathbb{R}^6\} = \mathbb{R}^{10}$, since $\dim \mathbb{R}^{10} = 10 > 6$.

Please record your answers on the title page.

3. Let $A = \begin{bmatrix} 1 & 1 & 0 & -5 \\ 2 & 1 & 3 & 2 \\ 3 & 1 & 3 & 4 \\ 1 & 0 & 0 & 2 \end{bmatrix}$. The dimension of the solution-space of $Ax = 0$ is:

A. 1
B. 2
C. 4
D. 0
E. 3
F. Infinite

$\dim \ker A = 4 - \operatorname{rank} A.$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\operatorname{rank} A = 3$. Hence, $\dim \ker A = 4 - 3 = 1$

4. For what value of e does the vector (5, 3, e) belong to the subspace of $\mathbf{R}^3$ spanned by (3, 2, 0) and (4, 2, 3)?

A. 3
B. 3/2
C. 1
D. 2
E. 1/2
F. 5/2

$$\left[\begin{array}{cc|c} 3 & 4 & 5 \\ 2 & 2 & 3 \\ 0 & 3 & e \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3/2 \\ 0 & 1 & 1/2 \\ 0 & 3 & e \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & e-3/2 \end{array}\right]$$

$\therefore$ in order that this system be consistent, we must have $e = \frac{3}{2}$

Please record your answers on the title page.

5. A basis for the solution space of the system

$$u - 2x + 3y + 4z = 0$$
$$-2u + 4x - 5y - 6z = 0$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ -2 & 4 & -5 & -6 & 0 \end{array}\right]$$

is:

A. { (0, 0, 0, 0) }
B. { (2, 1, 0, 0), (2, 0, -2, 1) }
C. { (1, 2, 0, 0) }
D. { (2, 0, -2, 1) }
E. { (2, 1, 0, 0), (1, -3, -4, 1) }
F. { (2, 1, 0, 0) }

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cccc} & s & & t \end{array} \\ \sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \end{array}$$

$\therefore$ $u = 2s + 2t$
$x = s$
$y = -2t$
$z = t$

$\therefore (u, x, y, z) = s(2, 1, 0, 0) + t(2, 0, -2, 1)$

The fundamental sol^ns are

$\therefore (2, 1, 0, 0)$ & $(2, 0, -2, 1)$.

6. Consider the linear system

$$\begin{array}{rcrcrcr} x & & & + & z & = & -1 \\ 2x & + & y & + & z & = & -1 \\ x & - & 2y & + & cz & = & d \end{array}$$

a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank A and rank$[\,A\,|\,b\,]$ for all values of c and d.

b) Find all c and d so that this system has
   i) a unique solution,
   ii) infinitely many solutions, and
   iii) no solutions.

c) In case (ii) above, give a geometric description of the set of solutions.

a) $[A\,|\,b] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 1 & -2 & c & d \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & c-1 & d+1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & c-3 & d+3 \end{array}\right]$ $\frac{1}{2}$

Hence, rank $A = \begin{cases} 2 & \text{if } c = 3 \\ 3 & \text{if } c \neq 3 \end{cases}$ 1 & rank $[A|b] = \begin{cases} 2 & \text{if } c = 3 \ \& \ d = -3 \\ 3 & \text{if } c = 3 \ \& \ d \neq -3 \\ 3 & \text{if } c \neq 3. \end{cases}$ $1\frac{1}{2}$

b) (i) A unique sol^n is obtained $\Leftrightarrow$ rank$A$ = rank$[A|b] = 3 \Leftrightarrow c \neq -3$ $\frac{1}{2}$

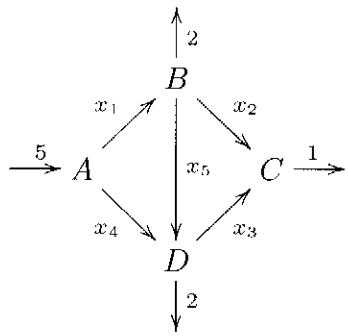
(ii) ∞ly many sol^ns occur $\Leftrightarrow$ rank$A$ = rank$[A|b] < 3 \Leftrightarrow c = 3$ & $d = -3$ $\frac{1}{2}$

(iii) no solutions are obtained $\Leftrightarrow$ rank$A$ < rank $[A|b] \Leftrightarrow c = 3$ & $d \neq -3$. $\frac{1}{2}$

c) (ii) : $[A|b] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad$ $x = -1 - s$, $y = 1 + s$, $z = s$ $\frac{1}{2}$; $s \in \mathbb{R}$.

This is the line through $(-1, 1, 0)$ with direction $(-1, 1, 1)$. 1 ($\frac{1}{2}$ if 1 or 2/3)

7. Consider the network of streets with intersections A, B, C and D below. The arrows indicate the direction of traffic flow along the one way streets, and the numbers refer to the number of cars observed to enter A or leave B,C and D during 1 minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Using Kirchoff's law, write down the linear system which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \dots, 5$. (Do not perform any operations on your equations: this is done for you in (b)!)

b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\begin{array}{c} \begin{array}{ccccccc} & & & s & t & & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Give the general solution. (Ignore the constraints at this point.)

c) Using (b) and the constraints from (a), find all possible traffic flows.

a)

	Flow in	=	Flow out
A:	5	$=$	$x_1 + x_4$
B:	x_1	$=$	$2 + x_2 + x_5$
C:	$x_2 + x_3$	$=$	1
D:	$x_4 + x_5$	$=$	$2 + x_3$

1

Constraints

$\frac{1}{2}$ $x_i \in \mathbb{Z}$ (number of cars)

$i = 1, .., 4$.

$\frac{1}{2}$ $x_i \geq 0$ (one-way streets)

b) $x_1 = 5 - s$ $\quad x_4 = s$
$x_2 = 3 - s - t$ $\quad x_5 = t$ $\quad$ 2 $\quad s, t \in \mathbb{R}$.
$x_3 = -2 + s + t$

c) From (a), $s, t \in \mathbb{Z}^+$ (nonnegative integers) $\Rightarrow$ all $x_i \in \mathbb{Z}$, & $x_4, x_5 \geq 0$)

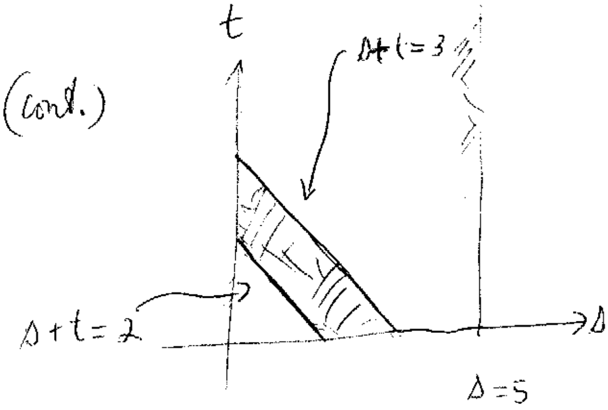
$x_1 \geq 0 \iff 5 - s \geq 0$ ie. $s \leq 5$

$x_2 \geq 0 \iff 3 - s - t \geq 0$ ie. $3 \geq s + t$

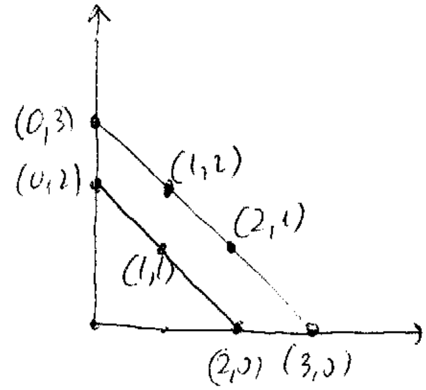
$x_3 \geq 0 \iff -2 + s + t \geq 0$ ie $s + t \geq 2$.

see over ...

7(c) (cont.)



$(s, t \in \mathbb{R})$



$(s, t \in \mathbb{Z})$

Hence there are 7 solutions, as follows ① justification

x_1	x_2	x_3	x_4 ($= s$)	x_5 ($= t$)
5	1	0	0	2
4	1	0	1	1
3	1	0	2	0
5	0	1	0	3
4	0	1	1	2
3	0	1	2	1
2	0	1	3	0

①

OR

i.e. using values of s, t s.t. ① $s, t \in \mathbb{Z}$
② $2 \le s+t \le 3$