B41A F96 Mid Term Solns

1. Which of the following are subspaces of R^3 ?

(1) $\{ (x, y, z) \mid 2x - y + 3z = 0 \}$
(2) $\{ (x, y, z) \mid xy = 0 \}$
(3) $\{ (x, y, z) \mid 2x = 5z \}$
(4) $\{ (x, y, z) \mid (x/2) = (y+3)/5 = 7z \}$

a. (1), (3) and (4)
b. (2) and (3)
c. (2) and (4)
d. (3) and (4)
e. (1) and (2)
f. (1) and (3)

(1) is a plane through the origin in $\mathbb{R}^3$ and $\therefore$ is a subspace. (normal is $(2,-1,3)$)

(2) is not $\{0\}$, a line or a plane through the origin in $\mathbb{R}^3$ and $\therefore$ isn't a subspace. (OR: it isn't closed under addn: $\underbrace{(1,0,0)}_{\in (2)} + \underbrace{(0,1,0)}_{\in (2)} = (1,1,0) \notin (2)$)

(3) is a plane through the origin in $\mathbb{R}^3$ (it has normal $(2,0,-5)$) and hence is a subspace.

(4) is a line, but does not pass through the origin, and so is not a subspace of $\mathbb{R}^3$.

2. Which one of the following is a basis for the subspace of $\mathbf{R}^3$
$G = \{ (x, y, z) \mid 2x - y + 3z = 0 \}$?

a. $\{(3, 0, -2)\}$
b. $\{(1, 0, 0), (1, 2, 0)\}$
c. $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$
d. $\{(1, 2, 0)\}$
e. $\{(1, 2, 0), (0, 3, 1)\}$
f. $\{(-3, 0, 2), (1, 0, 0)\}$

G is a plane through the origin in $\mathbb{R}^3$ and hence has dimension 2. So we need only consider b., e., & f.

b. The point $(1, 0, 0) \notin G$, so b. cannot be correct. This also eliminates f. from contention, so indeed e. is correct. (Note that $(1,2,0) \times (0,3,1)$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = (2, -1, 3),$$ which is the normal vector given.)

3. Consider the vector space

$$\mathbf{F}[0,2\pi] = \{f \mid f \text{ is a real-valued function defined on } [0,2\pi]\}$$

Recall that the zero of $\mathbf{F}[0,2\pi]$ is the function that has value 0 for all $x \in [0,2\pi]$.

2 a) Show that $\{x, \cos^2 x, \sin^2 x\}$ is linearly independent in $\mathbf{F}[0,2\pi]$.

1 b) Use De Moivre's theorem, i.e., $(\cos x + i\sin x)^n = \cos nx + i \sin nx$, to show that $\cos 2x = \cos^2 x - \sin^2 x$

1 c) Deduce from (b) that $\cos 2x \in \text{span}\{x, \cos^2 x, \sin^2 x\}$.

2 d) Is $\{\cos 2x, x, \cos^2 x, \sin^2 x\}$ linearly independent ? Explain.

$\frac{1}{2}$ for thisw/t $a=b=c=0$

a) Suppose $\underline{ax + b\cos^2 x + c\sin^2 x = 0}$ for all $x \in [0, 2\pi]$ 1

for $x = 0$, this implies $0 + b + 0 = 0$

$\frac{1}{2}$ $x = \pi/2$ " $a.\pi/2 + 0 + c = 0$

& $x = \pi$ " $a.\pi + b + 0 = 0.$

$\Rightarrow \underline{a = b = c = 0}$ $\frac{1}{2}$

$\therefore \{x, \cos^2 x, \sin^2 x\}$ is l.i.

b) $(\cos x + i \sin x)^2 \overset{\frac{1}{2}}{=} \cos^2 x - \sin^2 x + i\,(2\sin x \cos x) = \cos 2x + i \sin 2x,$
by De Moivre. Equating real parts yields $\cos^2 x - \sin^2 x = \cos 2x$. $\frac{1}{2}$

c) Since $\cos 2x = 0.x + \cos^2 x - \sin^2 x \in \text{span}\{x, \cos^2 x, \sin^2 x\}$,
we're done. 1

d) No, since by a theorem we know that $\{\cos 2x, x, \cos^2 x, \sin^2 x\}$ 1
is l.i. iff $\cos 2x \notin \text{span}\{x, \cos^2 x, \sin^2 x\}$. But the latter is true,
by c), hence $\{\cos 2x, x, \cos^2 x, \sin^2 x\}$ is l.d..

1 - correct reason

or: No, since by c)

$1.\cos 2x + 0x - 1.\cos^2 x + 1.\sin^2 x = 0$
for all $x \in [0, 2\pi]$.

4. a) Find the reduced row-echelon form of the augmented matrix of the linear system

$$\begin{array}{rcrcrcr} 2x & - & y & - & 4z & = & -2 \\ x & - & y & - & 2z & = & 1 \\ -x & - & 2y & + & 4z & = & -1 \end{array}$$

DO NOT ATTEMPT TO SOLVE THIS SYSTEM.

b) Suppose a linear system has augmented matrix

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & p \end{array}\right]}_{[\,A \,|\, b\,]} = [A|b]$$

Find all values of p and q for which this system has
(i) a unique solution,
(ii) infinitely many solutions, and
(iii) no solutions.

In case (ii) above, give a geometric description of the set of solutions. Is the set of solutions a subspace of $\mathbf{R}^3$? Explain.

4a) $\left[\begin{array}{ccc|c} 2 & -1 & -4 & -2 \\ 1 & -1 & -2 & 1 \\ -1 & -2 & 4 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 2 & -1 & -4 & -2 \\ 0 & -3 & +2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & -3 & +2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & +2 & -12 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & -11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 6 \end{array}\right]$

b) $\operatorname{rank}[A|b] = 3 \Leftrightarrow q \neq 0$. $\quad \therefore q \neq 0 \Leftrightarrow$ there's a unique solⁿ.
$\quad \| $
rank A

If $q = 0$,
rank $A = 1$, and rank $[A|b] = 2$ if $p \neq 0$ $\quad \therefore$ no solution iff $q = 0$ & $p \neq 0$

If $q = 0$ & $p = 0$, rank A = rank $[A|b] = 1 < 3 = \#$ vars $\therefore \infty$ly many solns $\Leftrightarrow q = 0$ & $p = 0$.

If $p = q = 0$, $[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ is the equation $x + y + z = 0$. The set of solutions here is a plane through 0 with normal $(1,1,1)$, which is a subspace of $\mathbb{R}^3$

5. a) A pet fanatic wishes to keep dogs, cats and weasels. They are to be fed a combination of three food types (A, B and C) each week, and their requirements (in kilograms per pet per week) are summarized below:

	A	B	C
Cats	2	2	3
Dogs	0	1	1
Weasels	2	1	2

The fanatic can only afford 12 kgs of A, 10 kgs of B and 16 kgs of C each week. Assuming that all the available food is eaten, write down a linear system, together with all constraints, which describes the possible combinations of cats, dogs and weasels that this lunatic can keep. DO NOT ATTEMPT TO SOLVE THIS SYSTEM.

b) Find all solutions of

$$\begin{array}{rcrcrcl} 2x & + & y & + & z & = & 14 \\ 3x & + & 3y & & & = & 15 \\ 5x & + & 4y & + & z & = & 29 \end{array}$$

such that x, y and z are integers with $x \geq 3, y \geq 0$ and $z \geq 2$.

5a) Let x = # cats, y = # dogs & z = # weasels the fanatic can keep. Then assuming all the food is eaten,

$$2x + 0y + 2z = 12 \quad \text{(food type A)}$$
$$2x + y + z = 10$$
$$\& \quad 3x + y + 2z = 16.$$

Moreover, x, y, z must be <u>positive</u> <u>integers</u>.

b) $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 14 \\ 3 & 3 & 0 & 15 \\ 5 & 4 & 1 & 29 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 4 \\ 0 & -1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 9 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \therefore \begin{array}{l} x = 9 - s \\ y = -4 + s \\ z = s \end{array};$

Now, $x, y, z \in \mathbb{Z} \Leftrightarrow s \in \mathbb{Z}$. Moreover, $x \geq 3 \Leftrightarrow s \leq 6$, $y \geq 0 \Leftrightarrow s \geq 4$, & $z \geq 2$ iff $s \geq 2$. In summary, $s \in \mathbb{Z}$, and $6 \geq s \geq 4$ i.e. $s = 4, 5$ or 6. $\therefore$ solutions are $(5, 0, 4)$, $(4, 1, 5)$, $(3, 2, 6)$.