MT3 ~~1996~~ 341A.F96

Andrew/Erik: please return these to me. Thanks.

1. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, then $\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix}$ is $= 4 \begin{vmatrix} g & a & d-2a \\ h & b & e-2b \\ i & c & f-2c \end{vmatrix}$ $2C_2+C_3 \to C_3$

a. 6
b. -12
c. -24
d. 12
e. 24
f. -6

$$= 4 \begin{vmatrix} g & a & d \\ h & b & e \\ i & c & f \end{vmatrix} = -4 \begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = +4 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$= 4 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ (since $\det A = \det A^t$)

$$= 12$$

2. Let A and B be n×n matrices, and x a scalar. Which three of the following statements could be false?

i) det (AB) = (det A) (det B) ✓ always true

× ii) det A + det B = det (A + B) — could be false: $\det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(0 + 0 \neq 1)$

× iii) det (xA) = x (det A) – only true if n=1, so it could be false, e.g. $\det\left(2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 4 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 2 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

iv) det (xA) = x^n det (A) ✓ always true

v) det (A^t) = det (A) ✓ always true

× vi) det (A^t) = - det (A) only true if $\det A = 0$; could be false, e.g. $\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq -1 \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a. (i), (iii), and (vi)
b. (ii), (iii), and (vi)
c. (i), (iii), and (v)
d. (i), (iv), and (vi)
e. (ii), (iv), and (v)
f. (ii), (iii), and (v)

3. Given the matrices

$$A = \begin{bmatrix} 7 & 5 & 1 \\ 2 & 0 & 5 \\ 7 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & -1 \\ 3 & 6 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 7 & 6 \\ 0 & 1 & 1 \\ 0 & 2 & -5 \end{bmatrix},$$

indicate which one of the following statements is true.

a. Only B is invertible
b. Only C is invertible
c. B and C are both invertible.
d. Only A is invertible
e. A and C are both invertible
f. A and B are both invertible

C is not invertible, since it has a column of zeros.

A is not invertible, since 2 rows are the same

$$\det B = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 9 & -1 \\ 3 & 6 & 8 \end{vmatrix} \overset{\text{row 1}}{=} 1 \cdot \begin{vmatrix} 9 & -1 \\ 6 & 8 \end{vmatrix} = 78 \neq 0 \therefore \text{ B is invertible}$$

4. If $A^{-1} = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ -1 & -2 & 5 \end{bmatrix}$, then $(A^t)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is:

$$(A^t)^{-1} = (A^{-1})^t = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ -4 & 3 & 5 \end{bmatrix}$$

a. $\begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$ b. $\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$ c. $\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

$$\therefore (A^{-1})^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

d. $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ e. $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ f. $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$, which one of the following statements is true for A^{-1} ?

a. The third row is [-1 -1 1]
b. The second column is $[0 \ \ 2 \ \ -1]^t$
c. The first row is [2 0 -1]
d. It does not exist.
e. Each of B, C, D and E is true.
f. The second row is [1 2 -1]

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{c|ccc} & 1 & 2 & -1 \\ I_3 & 2 & 0 & -1 \\ & -1 & -1 & 1 \end{array}\right]$$

6. Let A be an $n \times n$ matrix.

a) Give 4 different statements which are equivalent to the following:

A is invertible.

b) Give an example of a 2×2 matrix A such that $A^2 = 0$ but $A \neq 0$.

c) If A is a 2×2 matrix such that $A^7 = 0$, prove that A is not invertible. (Do not use a numerical example here.)

a) See your notes. $4 \times \frac{1}{2}$ each

1 b) Try $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $0 \neq A$, but $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

or $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ .... or $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ ..

2 c) Since $A^7 = 0$, $0 = \det(A^7) = (\det A)^7$. Hence

$\det A = 0$, so A is not invertible.

1 - $\det A = 0 \Rightarrow$ A not invertible

1 - showing $\det A = 0$.

(or any other correct proof.)

(Notice that it really doesn't matter that A is a 2×2 matrix here; the statement "$A^7 = 0 \Rightarrow \det A = 0$" is true for any size A. Moreover, the exponent "7" is not important either, any exponent will do. So the statement

"If $A^k = 0$ for some $k \in \mathbb{N}$, then A is not invertible"

is true.)

7. Let $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$.

a) Find $\det(A - tI_3)$. (*Hint:* Use the row operation $R_3 + R_2 \to R_2$.)

b) Check that if $t = -3$ or $t = 1$, then $(A - tI_3)$ is not invertible. (Don't work hard: use part (a) and a theorem.)

c) Find the general solution of $(A + 3I_3)x = 0$ and then use this to find a basis for the set of solutions $E = \{x \in \mathbf{R}^3 \mid (A + 3I_3)x = 0\}$ of this homogeneous system.

d) Give an equation for E and describe it geometrically.

a) $\begin{vmatrix} 5-t & 8 & 16 \\ 4 & 1-t & 8 \\ -4 & -4 & -11-t \end{vmatrix} \overset{(R_3+R_2\to R_2)}{=} \begin{vmatrix} 5-t & 8 & 16 \\ 0 & -3-t & -3-t \\ -4 & -4 & -11-t \end{vmatrix} = -(3+t)\begin{vmatrix} 5-t & 8 & 16 \\ 0 & 1 & 1 \\ -4 & -4 & -11-t \end{vmatrix}$

$\frac{1}{2}$ $\frac{1}{2}$ Some correct work

$\overset{(-C_2+C_3\to C_3)}{=} -(3+t)\begin{vmatrix} 5-t & 8 & 8 \\ 0 & 1 & 0 \\ -4 & -4 & -7-t \end{vmatrix} \overset{\text{row 2}}{=} -(3+t)\begin{vmatrix} 5-t & 8 \\ -4 & -7-t \end{vmatrix} = -(3+t)\{t^2+2t-35+32\}$

$= -(3+t)\{t^2+2t-3\} = -(t+3)^2(t-1) = p(t).$ $\frac{1}{2}$

b) From (a), $\det(A+3I) = p(-3) = 0$, so $A+3I$ is not invertible. Similarly, $\det(A-I) = p(1) = 0$, so $A-I$ is not invertible either. 1

c) $\begin{bmatrix} 8 & 8 & 16 & 0 \\ 4 & 4 & 8 & 0 \\ -4 & -4 & -8 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{ccc} & s & t \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$ $\therefore$ gen'l sol'n $\begin{array}{l} x = -s-2t \\ y = s \\ z = t \end{array}$ $s, t \in \mathbb{R}$. 1

$\therefore (x,y,z) = s(-1,1,0) + t(-2,0,1).$ (*) Hence $\{(-1,1,0),(-2,0,1)\}$ is a basis for E, since these 2 vectors are not parallel, and span E by (*). (only $\frac{1}{2}$ if no justification) 1

d) From (c) $x+y+2z=0$ $\frac{1}{2}$ is an eqn for E. It's a plane through O with normal $(1,1,2)$. 1