Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1341A Final Exam

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Family Name: Solutions

First Name: ____________________

Student number: ________________

1	F
2	A
3	C
4	A
5	F
6	D
sub-total	
7	
8	
9	
10	
11	
12	
Total	

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

0. This is a closed book exam.
1. You have three hours to complete this exam.
2. Questions 1–6 are multiple choice. These questions are worth **2** points each and no part marks will be given. Please record your answers in the space provided opposite.
3. Questions 7-12 require a complete solution, and are worth **6** points each, so spend your time accordingly. **The correct answer requires justification** written legibly and logically – you must convince me that your solution is correct. **You must answer these questions in the space provided.** Use the backs of pages if necessary.
4. Where it is possible to check your work, do so.
5. Good luck! Bonne chance!

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1. Find the value of the constant a for which the following system is inconsistent

$$\begin{aligned} x + y + z &= 1 \\ 2x + ay + 4z &= 1 \\ y - z &= 1 \end{aligned}$$

A. a = -4
B. a = -1
C. a = 2
D. a = -2
E. a = 1
F. a = 0

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & a & 4 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & a-2 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a & 1-a \end{array}\right]$$

$(2-a)(-1)+2 = a-2+2 = a$

$(2-a)(1)+(-1) = 2-a-1 = 1-a$

2. Which of $U = \{ (x, y, x\text{-}y) \mid x, y \in \mathbf{R} \}$, $V = \{ (x, y, x+y) \mid x, y \in \mathbf{R} \}$ and $W = \{ (x, y, xy) \mid x, y \in \mathbf{R} \}$ are subspaces of $\mathbf{R}^3$?

A. U and V only
B. V only
C. U and W only
D. W only
E. U only
F. V and W only

$U = \text{span}\{(1,0,1), (0,1,-1)\}$ ✓

$V = \text{span}\{(1,0,1), (0,1,1)\}$ ✓

$$\underset{\in W}{(1,0,0)} + \underset{\in W}{(0,1,0)} = \underset{\notin W}{(1,1,0)}$$

3. Which of the following is (are) a basis (bases) of $\mathbf{R}^3$?

V (1) { (1, 0, 1) (1, 4, 0) (-4, -4, 7) }
X (2) { (3, -1, 2) (5, 1, 1)}
X (3) { (2, 1, 3) (3, 1, -3) (1, 1, 9) }

A. None is a basis.
B. 2 and 3.
C. Only 1.
D. 1 and 3 .
E. Only 3.
F. 1 and 2.

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 4 & 0 \\ -4 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & -1 \\ -4 & -4 & 11 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ -4 & 11 \end{vmatrix} = 40 \neq 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & -3 \\ 1 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 5 & 2 & 0 \\ -5 & -2 & 0 \end{vmatrix} = 0 \quad \times$$

4. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$, what is the second row of A^{-1} ?

A. [-15/8, 1/2, 3/8]
B. [-15/4, 1, 3/4]
C. [13/8, -1/2, -1/8]
D. [-1/2, 1/2, 0]
E. [-15/8, 1/8, 3/8]
F. [-15, 4, 3]

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & +5/4 & 0 & -1/4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} & & & & * & \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & & * & \end{array}\right]$$

$-3/2 \cdot 5/4$

$-3/2 \cdot -1/4$

5. For which value(s) of x is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & x \end{bmatrix}$ invertible?

A. $x \neq 0$
B. $x = -1$
C. $x = 1$
D. $x \neq 1$
E. $x \neq \pm 1$
F. $x \neq -1$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & x \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & x \end{vmatrix} = x+1 \neq 0$$

$\Leftrightarrow x \neq -1$

6. A is an 8 by 6 matrix such that $Ax = 0$ has only the trivial solution, $x = 0$. Answer the following questions:

- What is the rank of A?
- Is $Ax = b$ consistent for all $b \in \mathbf{R}^8$?

$$\begin{array}{cc} & 6 \\ 8 & \left[\begin{array}{c|c} A & b \end{array}\right] \end{array}$$

A. 0, Yes
B. 8, No
C. 2, Yes
D. 6, No
E. 6, Yes
F. 8, Yes

$\dim \{x \mid Ax = 0\} + \text{rank } A = \#\text{cols of } A$

$\therefore\ 0 + \text{rank } A = 6$

$\therefore\ \text{rank } A = 6$

$[A \mid b]$ consistent $\forall$ all $b \in \mathbb{R}^8$

$\Leftrightarrow$ cols of A span $\mathbb{R}^8$. But

$\dim \text{span}\{\text{cols of } A\} \leq \#\text{col of } A = 6$

$\therefore$ NO.

7. Consider the linear system

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ x & + & 3y & + & 2kz & = & 0 \\ 2x & + & 4y & + & kz & = & 0 \end{array}$$

2 a) If A is the coefficient matrix of the system above, find rank A for all values of k.

1.5 b) Find all k so that this system has

i) a unique solution,

ii) infinitely many solutions, and

iii) no solutions.

2.5 c) In case (ii) above, give a geometric description of the set of solutions in each instance.

a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 2k & 0 \\ 2 & 4 & k & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2k-1 & 0 \\ 0 & 0 & k-2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & k-2 & 0 \end{array}\right]$ 1 $\therefore$ rank $A = \begin{cases} 2 & \text{if } k = 2 \\ 3 & \text{if } k \neq 2 \end{cases}$ 1

b) i) unique soln $\Leftrightarrow$ rank A = # vars $\Leftrightarrow$ $k \neq 2$. $\frac{1}{2}$

ii) ∞ly many solns $\Leftrightarrow$ rank $A < 3 \Leftrightarrow k = 2$ $\frac{1}{2}$

iii) a homog. system is always consistent. $\frac{1}{2}$

~~$\frac{1}{2}$, justification~~ some

c) $\underline{k=2}$: $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\frac{1}{2}$ $\therefore \begin{cases} x = 5s \\ y = -3s \\ z = s \end{cases}$ $\frac{1}{2}$, $\underline{s \in \mathbb{R}}$. 1

This is the line through O in $\mathbb{R}^3$ with direction $(5, -3, 1)$. $1\frac{1}{2}$

(plane – $0/1\frac{1}{2}$

(line with "normal" $1/1\frac{1}{2}$)

8. a) A pet fanatic, Pat, wishes to keep combination of dogs, cats and weasels. They are to be fed 3 food types, A, B and C, and their weekly requirements (in kgs per animal per week) are given in the table below:

	cats	dogs	weasels
A	2	0	2
B	2	1	1
C	3	1	2

Pat can only afford 12 kgs of A, 10 kgs of B and 16 kgs of C each week. Assuming that all the available food is eaten by Pat's furry friends, write down a linear system, together with all constraints, describing the possible combinations of cats, dogs and weasels that this lunatic can keep. Don't forget to define your variables and **DO NOT SOLVE THIS SYSTEM.**

b) Find all solutions of the following constrained linear system:

$$\begin{aligned} 2x + y + z &= 14 \\ 3x + 3y &= 15 \\ 5x + 4y + z &= 29 \end{aligned}$$

where x, y and z are integers, and $x \geq 3$, $y \geq 0$ and $\cancel{x}\ z \geq 2$.

$\boxed{3}$ a) Let x = # cats Pat can keep
y = # dogs "
z = # weasels " 1 If all food is eaten.

Type A : $2x + 2z = 12$
B : $2x + y + z = 10$
C : $3x + y + 2z = 16$ 1

$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 12 \\ 2 & 1 & 1 & 10 \\ 3 & 1 & 2 & 16 \end{array}\right]$

Constraints : Moreover, 1) $x, y, z \in \mathbb{Z}$ $\frac{1}{2}$
2) $x, y, z \geq 0$. $\frac{1}{2}$

b) $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 14 \\ 3 & 3 & 0 & 15 \\ 5 & 4 & 1 & 29 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 4 \\ 0 & -1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 9 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\frac{1}{2}$

$\therefore$ $x = 9 - s$
$y = -4 + s$
$z = s$ 1

Need $s \in \mathbb{Z}$ so that $x, y, z \in \mathbb{Z}$ $\frac{1}{2}$;
$x \geq 3 \iff 9 - s \geq 3 \iff s \leq 6$
$y \geq 0 \iff -4 + s \geq 0 \iff s \geq 4$ $\frac{1}{2}$
$z \geq 2 \iff s \geq 2$ ✓

$\frac{1}{2}$ $\therefore s = 4, 5$ or 6

$\therefore$ Soln $(x, y, z) =$
$(5, 0, 4), (4, 1, 5), (3, 2, 6)$
$(s = 4 \quad 5 \quad 6)$

1 – unique soln ($\in (\mathbb{Z}^+)^3$)

9. Let $W = \{(a, a-b, a+b) \mid a, b \in \mathbf{R}\}$.

1 a) By any method, show that W is a subspace of $\mathbb{R}^3$.

$2\frac{1}{2}$ b) Find a basis of W and give the dimension of W. (You must explain why your choice is a basis.)

$1\frac{1}{2}$ c) Give a geometric description of W.

1 d) Give an equation for W.

a) 1 $W = \text{Span}\{\overset{v_1}{(1,1,1)}, \overset{v_2}{(0,-1,1)}\}$ and so is a s.s. of $\mathbb{R}^3$.

b) $\{v_1, v_2\}$ spans W (from (a)) and v_1 & v_2 are not parallel, ~~so~~ ½
$\{v_1, v_2\}$ is l.i, ~~and~~ hence a basis of W. Hence, $\dim W = 2$.
Correct
1
1 justif.

c) W is the plane through the origin in $\mathbb{R}^3$ with
$1\frac{1}{2}$ normal $= v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = (2,-1,-1)$.
$(\frac{1}{2})$ $(\frac{1}{2})$ $(\frac{1}{2})$

d) From (c), $W = \{(x,y,z) \in \mathbb{R}^3 \mid (2,-1,-1)\cdot(x,y,z) = 0\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid 2x - y - z = 0\}$

$\therefore$ $2x - y - z = 0$ is an eqn for W. 1

b) – (-1) if no mention of l.i.

$(-\frac{1}{2})$ if mentioned, but no just given

10. Consider the vector space

$$\mathbf{F}[-1,1] = \{f \mid f \text{ is a real-valued function with domain } [-1,1]\},$$

together with the usual operations on functions. Recall that the zero of $\mathbf{F}[-1,1]$ is the function that has the value 0 for every $x \in [-1,1]$.

2 a) Prove that $\{1, x, x^2\}$ is linearly independent in $\mathbf{F}[-1,1]$.
(Hint: Consider $x = -1, 0$, and 1.)

1 b) If $f(x) = (2x+1)^2$, is $f \in \text{span}\{1, x, x^2\}$?

$1\frac{1}{2}$ c) Show that if $g(x) = \frac{1}{x+2}$, then $g \notin \text{span}\{1, x\}$

$1\frac{1}{2}$ d) What is the dimension of the subspace $U = \text{span}\{1, x, \frac{1}{x+2}\}$?

a) Suppose $a.1 + bx + cx^2 = 0$ for all $x \in [-1,1]$.

At $x = -1 \Rightarrow a - b + c = 0$

$0 \quad a = 0$

$1 \quad a + b + c = 0$

$\Rightarrow a = 0, b = 0, c = 0$ 1

$\therefore \{1, x, x^2\}$ are l.i. in $F[-1,1]$

b) Since $f(x) = 4x^2 + 4x + 1 = 1.1 + 4.x + 4.x^2$, 1
$f \in \text{span}\{1, x, x^2\}$.

c) Suppose $g = a1 + b.x$ for all $x \in [-1,1)$. Then at $\frac{1}{2}$ + 1 correct arg.

$x =$

d) $1\frac{1}{2}$ - correct dim.

11. a) Let A be a real $n \times n$ matrix. Give 3 different statements which are equivalent to:

3×1

$$\det A \neq 0. \quad \text{(see notes)}$$

b) Say whether the following statements are TRUE or FALSE. No justification is necessary!

i) If is A a real $n \times n$ matrix, then A is invertible iff 0 is not an eigenvalue of A. TRUE

3×1 ii) A 3×4 matrix can have linearly independent columns. FALSE

iii) The solution space of a homogeneous system of 2 equations in 6 unknowns can have dimension exactly 3. FALSE

i) A is invertible $\Leftrightarrow \det A \neq 0 \Leftrightarrow \det(A - 0I) \neq 0 \Leftrightarrow 0$ is not an eval. of A.

ii) If A is 3×4, then $\operatorname{rank} A \leq 3$, so the cols of A are l.d.

iii) If A is 2×6, then $\dim\{x \mid Ax = 0\} = 6 - \operatorname{rank} A$.

But $\operatorname{rank} A \leq 2$, so $\dim\{x \mid Ax = 0\} \geq 4$.

12. Let $A = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. The eigenvalues of A are 0 and 6.

a) Find a basis of $E_0 = \{x \in \mathbf{R}^3 \mid Ax = 0\}$. 1

b) Find a basis of $E_6 = \{x \in \mathbf{R}^3 \mid (A - 6I)x = 0\}$. $1\frac{1}{2}$

c) Show that the set consisting of all vectors in the bases for E_0 and E_6 is a basis for $\mathbf{R}^3$. 1

d) If possible, find an invertible matrix P such that $P^{-1}AP = D$ is diagonal and give this diagonal matrix D. $\frac{1}{2}$ 1 1

a) $\left[\begin{array}{ccc|c} 5 & -1 & 2 & 0 \\ -1 & 5 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ so $x = -t/2$, $y = -t/2$, $z = t$ (1)

So $E_0 = \text{span}\{(-1/2, -1/2, 1)\}$; basis $= \{(-1/2, -1/2, 1)\}$

b) $[A - 6I \mid 0] = \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\therefore$ $x = -s + 2t$, $y = s$, $z = t$

$\therefore E_6 = \text{span}\{(-1,1,0), (2,0,1)\}$. These 2 vectors are l.i. (not $\parallel$)

so $\{(-1,1,0), (2,0,1)\}$ is a basis for E_6.

c) $\begin{vmatrix} -1/2 & -1/2 & 1 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -2\frac{1}{2} & -1/2 & 0 \\ -1 & -1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -2.5 & -.5 \\ -1 & -1 \end{vmatrix} = 2 \neq 0$

$\therefore$ they are a basis of $\mathbb{R}^3$.

d) $\therefore P = \begin{bmatrix} -1/2 & -1 & 2 \\ -1/2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is invertible

so $P^{-1}AP = D = \begin{bmatrix} 0 & & \\ & 6 & \\ & & 6 \end{bmatrix}$