Mat 1341A Mid Term ('98) Solutions

1. For the following system of 2 equations in the 3 unknowns x, y and z,

$$\begin{aligned} x + 5y \quad &= 6 \\ z &= 1 \end{aligned}$$

A. The system is inconsistent.
B. (0, 0, 0) is a solution.
C. (6s-5, s, 1) is a solution for any value of s.
D. $\left(6, \frac{6}{5}, 1\right)$ is a solution.
E. (6-5s, s, 1) is a solution for any value of s.
F. (6, 1, 0) is a solution.

Solving the first eqn, we have

$$x = 6-5y,$$

so any solution is of the form

$(6-5y, y, 1)$ for any value of y.

2. The vector $v = (1,0,-1)$ can be written as $v = c_1v_1 + c_2v_2 + c_3v_3$, where $\{v_1, v_2, v_3\}$ is the orthonormal basis with

$$v_1 = \frac{\sqrt{3}}{3}(1, 1, 1), \qquad v_2 = \frac{\sqrt{6}}{6}(1, 1, -2), \qquad \text{and} \qquad v_3 = \frac{\sqrt{2}}{2}(1, -1, 0),$$

Then, $(c_1, c_2, c_3) =$

A. $\left(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{3}\right)$

B. $\left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{3}\right)$

C. $\left(0, \frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$

D. $\left(1, \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}\right)$

E. $\left(\frac{4\sqrt{3}}{3}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{3}\right)$

F. $\left(4, \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}\right)$

Since $\{v_1, v_2, v_3\}$ is orthonormal,

$$c_1 = \frac{v \cdot v_1}{v_1 \cdot v_1} = \frac{v \cdot v_1}{1} = 0$$

$$c_2 = \frac{v \cdot v_2}{v_2 \cdot v_2} = \frac{v \cdot v_2}{1} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}$$

(we can see that C is correct now...

$$c_3 = v \cdot v_3 = \frac{\sqrt{2}}{2}$$

3. Consider the vector space

$$\mathbf{F}[0,2\pi] = \{f \mid f \text{ is a real-valued function defined on } [0,2\pi]\}$$

Recall that the zero of $\mathbf{F}[0,2\pi]$ is the function that has the value 0 for all $x \in [0,2\pi]$.

a) Show that $\{1, \cos 2x, \sin 2x\}$ is linearly independent in $\mathbf{F}[0,2\pi]$.

b) If $z = \cos x + i \sin x$, use a well-known trigonometric identity to show that $|z| = 1$.

c) Use (b) and De Moivre's theorem, i.e., $(\cos x + i\sin x)^n = \cos nx + i \sin nx$, to show that $\cos 2x = 2\cos^2 x - 1$

d) Deduce from (c) that $\cos^2 x \in \text{span}\{1, \cos 2x, \sin 2x\}$.

e) Is $\{\cos^2 x, 1, \cos 2x, \sin 2x\}$ linearly independent in $\mathbf{F}[0,2\pi]$? Explain.

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a) Suppose $a.1 + b\cos 2x + c.\sin 2x = 0$ for all $x \in [0, 2\pi]$. Then,

for $x = 0$, we have $a + b = 0$

$x = \pi/4$ " $a + c = 0$ (1) ← Same system that works (to get $a=b=c=0$)

$x = \pi/2$ " $a - b = 0$

The 1st and 3rd eqns imply that $a=b=0$, which, using the 2nd eqn, implies $c=0$. Hence $a=b=c=0$, so $\{1, \cos 2x, \sin 2x\}$ is l.i.

(1/2) (even if the system doesn't guarantee this)

(1/2) b) $|\cos x + i\sin x|^2 = \cos^2 x + \sin^2 x = 1 \quad \therefore |\cos x + i \sin x| = 1.$

(1) c) Let $n=2$ to see that $(\cos x + i\sin x)^2 = \cos 2x + i \sin 2x$. The LHS is $\cos^2 x - \sin^2 x + i(2\sin x \cos x)$. Equating real parts yields $\cos^2 x - \sin^2 x = \cos 2x$. But from (b), $\sin^2 x = 1 - \cos^2 x$, so

$\cos^2 x - (1 - \cos^2 x) = \cos 2x$ i.e. $\cos 2x = 2\cos^2 x - 1$.

(1) d) From (c), $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x = \frac{1}{2}.1 + \frac{1}{2}\cdot\cos 2x \in \text{span}\{1, \cos 2x, \sin 2x\}$

(1) e) No, because $\cos^2 x \in \text{span}\{1, \cos 2x, \sin 2x\}$, as was shown in d)

4. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x + z = 0\}$

a) Give a complete geometric description of W.

b) Is W a subspace of $\mathbf{R}^3$? Give reasons for your answer.

c) Find a basis for W (you must check at least one of the two conditions to be satisfied) and hence find the dimension of W.

d) Is your chosen basis orthogonal? Justify your answer.

(need all 3 to get 1; $x < 3 - \frac{1}{2}$

1 a) W is the plane through the origin in $\mathbb{R}^3$ with normal $(1,0,1)$.

1 b) Yes, because every plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$

c) $W = \{(x,y,z) \in \mathbb{R}^3 \mid x+z=0\}$ $\quad (x+z=0 \iff x=-z.)$

$= \{(-z, y, z) \in \mathbb{R}^3 \mid y, z \in \mathbb{R}\}$

$= \{z(-1,0,1) + y(0,1,0) \mid y, z \in \mathbb{R}\}$

$= \text{span}\{(-1,0,1), (0,1,0)\}$

Thus $\{(-1,0,1), (0,1,0)\}$ spans W. Since these 2 vectors aren't parallel, they are l.i., and hence form a basis of W. 1 } + reason = 1

Thus, $\dim W = 2$ 1 (2 vectors in the basis)

1 d) Well, $(-1,0,1) \cdot (0,1,0) = 0$, and each vector in it is non-zero, so yes, it is an orthogonal basis of W.

5. In each case, give an explicit example of:

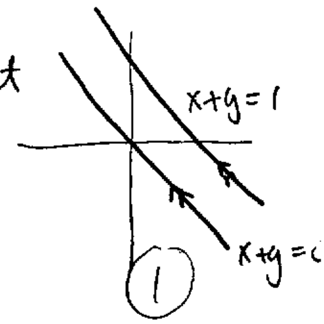
2 a) a linear system of 2 equations in 2 unknowns with no solutions , with a sketch

2 b) a linear system of 2 equations in 3 unknowns with at least 2 different solutions. (You must explicity give two different solutions.)

2 c) any linear system with a unique solution, which must be given.

a) e.g. $x + y = 0$
$x + y = 1$
1

is clearly an inconsistent system.
[(there isn't a unique answer anywhere here)]



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b) e.g. $x + y = 0$
$y + z = 0$
1

has solutions $(0,0,0)$ 1 and $(1,-1,1)$, for example

(the 2 planes intersect in a line through the origin)

c) eg. $x = 1$ 2 (one eqn, one unknown)

or e.g. $x + y = 0$
$x - y = 0$
1

(unique soln is $x = y = 0$)
1