MT1 F98

1. Find all real constants a so that

W= { f ∈ **F**(**R**) | s.t. the second derivative f'' exists, and f'' + f = a }

is a subspace of **F**(**R**).

- A. a < 0
- B. a = 0
- C. a = 1
- D. a > 0
- E. Any a ∈ **R**
- F. a = − 1

In order for the zero function to be in W, we must have

$0''(t) + 0(t) = a$ for all $t \in \mathbb{R}$.

However, $0''(t) = 0(t) = 0$, for all $t \in \mathbb{R}$, so we must have $a = 0$. Hence, B. is the only possible choice.

2. Which of the following are not vector spaces when the usual operations are used?

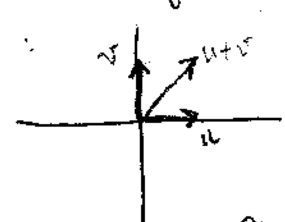
S.S. ✓ (1) { (x,y,z)∈ $\mathbf{R}^3$ | z = 0} is a plane through 0, so is a s.s. of $\mathbb{R}^3$, so is a v.s.

X (2) { (x,y)∈ $\mathbf{R}^2$ | x = 0 or y = 0} = U

✓ (3) { f ∈ **F**[0,1] | 2f(0) = f(1)} = V

- A. (3) only
- B. (2) and (3)
- C. (1) and (3)
- D. (1) and (2)
- E. (1) only
- F. (2) only

is the union of the x-axis & the y-axis :



This is not closed under addition : $\underset{u}{(1,0)} + \underset{u}{(0,1)} = (1,1) \notin U$.

this is a v.s. (using the usual operations) iff it is a subspace of F[0,1]: Since $2\,0(0) = 0 = 0(1)$, the zero function belongs to V.

If $f, g \in V$, then $2(f+g)(0) = 2f(0) + 2g(0) = f(1) + g(1) = (f+g)(1)$, so $f+g \in V$

If $f \in V$ and $a \in \mathbb{R}$, then $2(af)(0) = 2af(0) = a.f(1) = (af)(1)$, so $af \in V$.

Hence V is a s.s. of F[0,1], and so is a v.s.

3. Which of the following are subspaces of $\mathbf{R}^3$?

S.S.

✓ $U = \{ (x, x, x) \mid x \in \mathbf{R}\}$ = span$\{(1,1,1)\}$, so is a s.s. (its the line through O with direction (1,1,1).

X $V = \{ (x, y, z) \mid x^2 - y^2 = 0, z \in \mathbf{R}\}$

✓ $W = \{ (x, 2x, 3x) \mid x \in \mathbf{R}\}$

A. U and V only
B. W only
C. U only
D. U and W only
E. V only
F. V and W only

V is not closed under addition: $(1,1,0)$, $(1,-1,0) \in V$ but $(1,1,0) + (1,-1,0) = (2,0,0) \notin V$.

$W = \text{span}\{(1,2,3)\}$, and so is a s.s.

4. Which of the following are subspaces of $\mathbf{R}^3$?

S.S.

✓ (1) $\{ (x, y, z) \mid 2x - y + 3z = 0 \}$
X (2) $\{ (x, y, z) \mid xy = 0 \}$
✓ (3) $\{ (x, y, z) \mid 2x = 5z \}$
X (4) $\{ (x, y, z) \mid (x/2) = (y+3)/5 = 7z \}$

A. (2) and (3)
B. (1) and (2)
C. (1), (3) and (4)
D. (3) and (4)
E. (2) and (4)
F. (1) and (3)

(1) is a plane through O (with normal $(2,-1,3)$), so is a s.s. of $\mathbb{R}^3$

(2) is not closed under addition: $(1,0,0)$ and $(0,1,0)$ belong to (2), but $(1,0,0)+(0,1,0) = (1,1,0)$ does not.

(3) is the plane through the origin with normal $(2,0,-5)$, so is a s.s. of $\mathbb{R}^3$

(4) does not contain $(0,0,0)$, so is not a s.s. of $\mathbb{R}^3$

5. Which of the following is a subspace of $\mathbf{R}^4$?

A. None of the ~~above~~. below.
B. $\{(a, b, c, d)\} \mid a + d = 0\}$
C. $\{(a, b, c, d)\} \mid ad = 1\}$
D. $\{(a, b, c, d)\} \mid ad - bc = 0\}$
E. $\{(a, b, c, d)\} \mid$ a, b, c, d are integers.$\}$
F. $\{(a, b, c, d)\} \mid a = 1\}$

✓ B. $= \{(a,b,c,-a) \mid a,b,c \in \mathbb{R}\} = \text{span}\{(1,0,0,-1),(0,1,0,0),(0,0,1,0)\}$

and so is a s.s. of $\mathbb{R}^4$

✗ C. does not contain $(0,0,0,0)$.

✗ D. is not closed under addition, since $(1,0,0,0)$ and $(0,0,0,1)$ belong to D but their sum, $(1,0,0,1)$, does not.

✗ E. is not closed under multiplication by all real scalars: e.g. $(1,0,0,0) \in E$ but $\frac{1}{2}(1,0,0,0) = (\frac{1}{2},0,0,0) \notin E$.

✗ F. does not contain $(0,0,0,0)$.

6. Let $U = \mathrm{span}\{\overset{u_1}{(1, 0, 0)}, \overset{u_2}{(1, 1, 0)}\}$.

$1\frac{1}{2}$ (a) Using the subspace test, carefully prove that U is a subspace of $\mathbf{R}^3$.

$1\frac{1}{2}$ (b) Give a geometric description of U and find an equation for U.

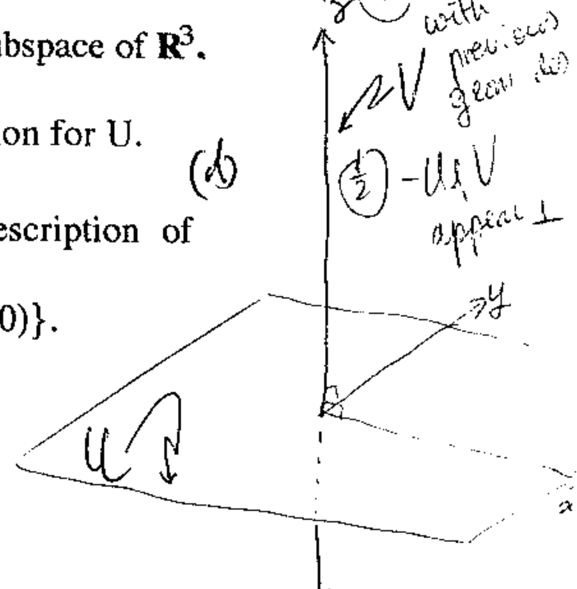
2 (c) Compute $(1, 0, 0) \times (1, 1, 0)$ and give a geometric description of

$$V = \mathrm{span}\{(1, 0, 0) \times (1, 1, 0)\}.$$

Is V a subspace of $\mathbf{R}^3$? Give reasons for your answer.

1 (d) Sketch U and V in $\mathbf{R}^3$.

(d)



(a) $0 = (0,0,0) = 0 u_1 + 0 \cdot u_2$, so $0 \in U$. (1/2)

"+" If $u = c_1 u_1 + c_2 u_2$ and $u' = c_1' u_1 + c_2' u_2$ belong to U, then
$u + u' = (c_1 + c_1') u_1 + (c_2 + c_2') u_2$ clearly belongs to U. (1/2)

"·" If $u = c_1 u + c_2 u_2 \in U$ and $a \in \mathbb{R}$, then
$au = (ac_1) u_1 + (ac_2) u_2$ clearly belongs to U. (1/2)

Hence U is a subspace of $\mathbb{R}^3$.

(b) U is the plane through 0 with normal $n = u_1 \times u_2 =$
$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = (0,0,1)$. (i.e. U is the x-y plane). An equation for U is thus $z = 0$. (1/2)

(1/2) (1/2 if only the thing is missing)

(c) $u_1 \times u_2 = (0,0,1)$ - done in (b). (1/2) Thus $V = \mathrm{span}\{(0,0,1)\}$ is the line through 0 with direction $(0,0,1)$ (1/2) (i.e. V is the z-axis). V is a subspace of $\mathbb{R}^3$, since all lines in $\mathbb{R}^3$ through 0 are subspaces (or: V is a s.s. since it is the span of a set of vectors, which is always a subspace.) (1)

(d) see above right ↗.

7. Let $u = (1, 0, 1) \in \mathbf{R}^3$.

2 (a) Show that for $v = (x, y, z) \in \mathbf{R}^3$,

$$v - \text{proj}_u v = \left(\frac{x - z}{2}, y, \frac{z - x}{2}\right)$$

1 (b) Use (a) to write $v - \text{proj}_u v$ as a linear combination of $(1, 0, -1)$ and $(0, 1, 0)$.

3 (c) Let

$$W = \{v - \text{proj}_u v \mid v \in \mathbf{R}^3\}$$
and
$$V = \text{span}\,\{(1, 0, -1), (0, 1, 0)\}.$$

Give a geometric description of W.
What is the relationship between the subspaces W and V?

3 (d) (Bonus) Show that a vector $w \in \mathbf{R}^3$ belongs to W if and only if $w.u = 0$.

(a) $(x,y,z) - \text{proj}_{(1,0,1)}(x,y,z) = (x,y,z) - \frac{(1,0,1)\cdot(x,y,z)}{\|(1,0,1)\|^2}(1,0,1)$
$= (x,y,z) - \frac{(x+z)}{2}(1,0,1) = \left(\frac{x-z}{2}, y, \frac{z-x}{2}\right).$ 2

b) If $v = (x,y,z)$, by (a), $v - \text{proj}_u v = \left(\frac{x-z}{2}, y, \frac{z-x}{2}\right)$

$= \left(\frac{x-z}{2}\right)(1,0,-1) + y(0,1,0)$, which is a l.c. of the required form. 1

(c) Part (b) shows that $W = \{(\frac{x-z}{2})(1,0,-1) + y(0,1,0) \mid (x,y,z) \in \mathbb{R}^3\}$.
Since x and z are arbitrary, so is $\frac{x-z}{2}$, so W is in fact the set of all linear combinations of $(1,0,-1)$ and $(0,1,0)$. That is, $W = \text{span}\{(1,0,-1), (0,1,0)\}$, which is just V. So $W = V$, $1\frac{1}{2}$
which is the plane through O with normal $(1,0,-1)\times(0,1,0) = (1,0,1)$.
i.e. $W = V$ is the plane through O normal to u.
$\frac{1}{2}$ + 1 for justification.

(d) see over!

(d) We must show 2 things: A. $w \in W \Rightarrow w \cdot u = 0$ and B. $w \cdot u = 0 \Rightarrow w \in W$. $\frac{1}{2}$

(be demanding here!)

A. Suppose $w = v - \mathrm{proj}_u v = \left(\frac{x-z}{2}, y, \frac{z-x}{2}\right)$. Then

1 $w \cdot (1,0,1) = \frac{x-z}{2} + \frac{z-x}{2} = 0$. Thus, $w \in W \Rightarrow w \cdot u = 0$.

B. Suppose $w = (x,y,z)$ satisfies $w \cdot u = 0$. Then, $x+z=0$, or $x=-z$. The y-component of w is unrestricted, so $w = (x, y, -x)$ for some $x, y \in \mathbb{R}$. Thus,

$1\frac{1}{2}$

$$w = x(1,0,-1) + y(0,1,0),$$ which is a linear combination of $(1,0,-1)$ and $(0,1,0)$, and so belongs to V. But we showed in part (c) that $V = W$, so w belongs to W.

Hence, we've shown that $w \cdot u = 0 \Rightarrow w \in W$.