1. The coefficient matrix A in a homogeneous system of 12 equations in 16 unknowns is known to have rank 6. How many parameters are there in the general solution?

$$\begin{array}{cc} & 16 \\ 12 & [A|0] \end{array} \quad \because \text{rank } A = 6,$$

A. 10
B. none
C. 6
D. 12
E. 4
F. 16

$$\# \text{ parameters} = n - r = 16 - 6 = 10$$

2. For what value of α does the vector $\overset{u}{(5, 3, \alpha)}$ belong to the subspace of $\mathbf{R}^3$ spanned by $\underset{v}{(3, 2, 0)}$ and $\underset{w}{(1, 0, 3)}$?

A. 2
B. 1
C. 5/2
D. 3/2
E. 1/2
F. 3

$$[v\,w|u] \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 2 & 3 \\ 3 & 0 & \alpha \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 2 & 3 \\ 0 & -9 & \alpha-15 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & \underbrace{\alpha-15+\frac{27}{2}}_{\alpha - 3/2} \end{array}\right]$$

The vector $u \in \text{span}\{v, w\}$ iff the system $[v\ w | u]$ is consistent. In order that the system be consistent, we need $\alpha - \frac{3}{2} = 0$, so $\alpha = 3/2$

(or, compute $u \cdot v \times w$ and set it equal to 0:

$v \times w = \begin{vmatrix} i & j & k \\ 3 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} = (6, -9, -2)$; so $u \cdot v \times w = 30 - 27 - 2\alpha = 0$

$\Leftrightarrow \alpha = \frac{3}{2}$)

Please record your answers on the title page.

3. If $A = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ is the coefficient matrix of a homogeneous linear system, the dimension of the solution space of that system is:

A. 3
B. 4
C. 2
D. Infinite
E. 1
F. 0

$$\begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank $A = 3$. Hence

dim sol^n space $= 4 - 3 = 1$.

4. For what value of α is $\underset{S}{=} \{\underset{u}{(1, 1, 1)}, \underset{v}{(1, 0, 2)}, \underset{w}{(1, \alpha, 1)}\}$ linearly dependent?

A. -1
B. 0
C. 1
D. -2
E. -1/2
F. 2

S is linearly dependent iff the system $[u \ v \ w \mid 0]$ has ∞ly many solutions, i.e. rank $[u \ v \ w] < 3$.

$$[u \ v \ w] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & \alpha \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & \alpha-1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha-1 \end{bmatrix},$$ which

has rank $< 3 \iff \alpha = 1$.

5. Which of the following statements is true for the following system?

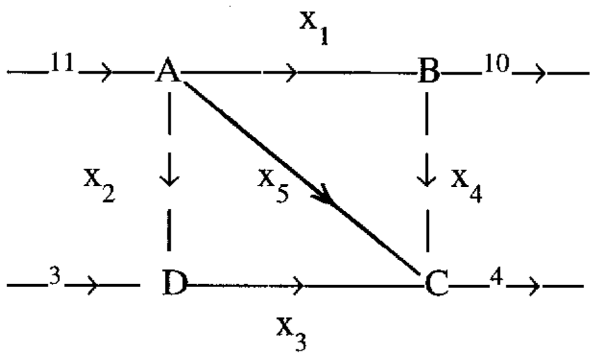
$$\begin{aligned} x + y \quad\quad &= 5 \\ 2x + y + z &= 13 \\ 4x + 3y + z &= 23 \end{aligned}$$

A. It is inconsistent
B. It has the unique solution (5, 0, 3).
C. It has infinitely many solutions with 2 parameters.
D. It has the trivial solution.
E. It has infinitely many solutions with 1 parameter.
F. It has the unique solution (8, −3, 0).

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 2 & 1 & 1 & 13 \\ 4 & 3 & 1 & 23 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, rank A = rank $[A|b] = 2 < 3 = \#$ vars, so this system has ∞ly many solutions with 1 $(= 3-2)$ parameter.

6. Consider the diagram of streets below. The arrows indicate the direction of traffic flow along the <u>one–way</u> streets, and the numbers refer to the number of cars observed to enter (in the case of A and D) or leave (in the case of B and C) the intersections during a period of 10 minutes. Each x_i denotes the (unknown) number of cars which passed along the indicated streets during the same period.



a) Using Kirchoff's law, write down a system of linear equations in the x_i, i=1,…,5, together with all constraints, which describe the the traffic flow.

b) If $\begin{array}{ccccc} & & & s & t \\ \end{array}$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & 0 & 10 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
is the result of row–reduction on the linear system of (a)

find the general solution, **ignoring the constraints.**

c) If AD had been closed to traffic because of roadwork, find all the possible values of the x_i, i=1,…,5.

a)

Intersection	# cars in	=	# cars out.
A	11	=	$x_1 + x_2 + x_5$
B	x_1	=	$10 + x_4$
C	$x_3 + x_4 + x_5$	=	4
D	$x_2 + 3$	=	x_3

1

Constraints: since each x_i represents the # of cars on one-way streets, $x_i \in \mathbb{Z}$ and $x_i \geq 0$ for $i = 1, \ldots, 5$.

b) $x_1 = 10 + s$ $\quad x_4 = s$
$x_2 = 1 - s - t$ $\quad x_5 = t$ $\quad ; s, t \in \mathbb{R}$.
$x_3 = 4 - s - t$

1

c) AD is closed $\iff x_2 = 1-s-t = 0$. Hence $s+t=1$.

Imposing the constraints, we find

i) $x_i \in \mathbb{Z} \iff s, t \in \mathbb{Z}$

& ii) $x_1 \geq 0 \iff 10+s \geq 0 \iff s \geq -10$

$x_2 \geq 0$ ✓ $(x_2 = 0)$

$x_3 \geq 0 \iff 4-s-t \geq 0 \iff s+t \leq 4$ ✓ $(s+t=1)$

$x_4 \geq 0 \iff s \geq 0$

$x_5 \geq 0 \iff t \geq 0$

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

Hence, $s \geq 0,\ t \geq 0$, $s+t=1$ and $s, t \in \mathbb{Z}$. Thus, the only possibilities are $(s,t) = (0,1)$ or $(1,0)$. $\frac{1}{2}$

$$\left.\begin{array}{l} \text{For } s=0,\ t=1:\ x_1=10,\ x_2=0,\ x_3=3,\ x_4=0,\ x_5=1 \\ \phantom{\text{For }} s=1,\ t=0:\ x_1=11,\ x_2=0,\ x_3=3,\ x_4=1,\ x_5=0. \end{array}\right\} 1$$

7. Consider the linear system:

$$\begin{aligned} x + y + 3z &= 2 \\ x + 2y + 5z &= 1 \\ 2x + 3y + az &= b \end{aligned}$$

a) Compute rank A and rank[A | B] for all values of a and b, where A and [A | B] are, respectively, the coefficient matrix and the augmented matrix of the system.

b) Find all values of a and b for which this system has

(i) a unique solution,
(ii) infinitely many solutions
and (iii) no solutions.

c) In the case (ii), give a geometric description of the set of solutions. Is it a subspace of $\mathbb{R}^3$?

a) $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 5 & 1 \\ 2 & 3 & a & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & a-6 & b-4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & a-8 & b-3 \end{array}\right]$ $\frac{1}{2}$

Hence, $\operatorname{rank} A = \begin{cases} 2 & \text{if } a = 8 \ \ \frac{1}{2} \\ 3 & \text{if } a \neq 8 \ \ \frac{1}{2} \end{cases}$ & $\operatorname{rank}[A|B] = \begin{cases} 2 & \text{if } a=8 \ \& \ b=3 \\ 3 & \text{if } a=8 \text{ and } b\neq 3 \\ 3 & \text{if } a \neq 8 \end{cases}$ $1\frac{1}{2}$

b) Using (a), we find that

(i) there's a unique solⁿ iff $\operatorname{rank} A = \operatorname{rank}[A|B] = \#\text{vars.} = 3$

i.e. iff $a \neq 8$ $\frac{1}{2}$ (b is not restricted)

(ii) ∞'ly many solns exist iff $\operatorname{rank} A = \operatorname{rank}[A|B] = 2 < 3 = \#\text{vars}$,

i.e. iff $a = 8$ and $b = 3$

(iii) the system is inconsistent $\Leftrightarrow$ $\operatorname{rank} A = 2 < 3 = \operatorname{rank}[A|B]$

i.e. iff $a = 8$ and $b \neq 3$.

c) $a=8,\ b=3 \Rightarrow [A|B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Hence the general solution is $\left.\begin{array}{l} x = 3 - s \\ y = -1 - 2s \\ z = s \end{array}\right\}$, $s \in \mathbb{R}$. $\frac{1}{2}$

This set of solⁿ is thus the line through $(3,-1,0)$ in $\mathbb{R}^3$ with direction $(-1,-2,1)$. $\frac{1}{2}$

It is not a subspace of $\mathbb{R}^3$, since $(0,0,0)$ is not a solution to the system in this case. $\frac{1}{2}$