1. Let $A = \begin{bmatrix} a & -1 \\ 2 & b \end{bmatrix}$. Find all (a, b) for which $A^2 = 0$.

A. $\pm(1, -1)$
B. $\pm\left(\sqrt{3}, -\sqrt{3}\right)$
C. $\pm\left(\sqrt{2}, \sqrt{2}\right)$
D. $\pm(1, 1)$
E. $\pm\left(\sqrt{3}, \sqrt{3}\right)$
F. $\pm\left(\sqrt{2}, -\sqrt{2}\right)$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A^2 = \begin{bmatrix} a^2-2 & -a-b \\ 2a+2b & -2+b^2 \end{bmatrix}$$

$\Leftrightarrow a^2 = 2 = b^2$ and $a = -b$

Hence $a = \pm\sqrt{2}$ and $b = \mp\sqrt{2}$

2. If A is the 2×2 matrix such that $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, compute A^2.

A. $\begin{bmatrix} 31 & -8 \\ -12 & 7 \end{bmatrix}$ B. $\begin{bmatrix} 29 & -13 \\ -13 & 10 \end{bmatrix}$ C. $\begin{bmatrix} -10 & 4 \\ 6 & 2 \end{bmatrix}$

D. $\begin{bmatrix} 30 & -8 \\ -12 & 7 \end{bmatrix}$ E. $\begin{bmatrix} 19 & 8 \\ -12 & 5 \end{bmatrix}$ F. $\begin{bmatrix} 25 & 4 \\ 9 & 1 \end{bmatrix}$

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = 1st col of A; $A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ = 2nd col of A $\therefore A = \begin{bmatrix} -5 & 2 \\ 3 & 1 \end{bmatrix}$; so

$$A^2 = \begin{bmatrix} -5 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 31 & -8 \\ -12 & 7 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find the main diagonal of A^{-1}.

A. (0, -1, 0)
B. (-1, 0, -1)
C. (-1, 0, 0)
D. (0, 0, 1)
E. (-1, -1, 1)
F. (0, 1, 0)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

4. The (2, 1)-entry of the product $\begin{bmatrix} 3 & 7 & -2 \\ 3 & 4 & 7 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} -3 & 9 & 6 \\ -7 & -8 & 4 \\ 1 & 0 & 5 \end{bmatrix}$ is:

A. 10
B. -25
C. -45
D. -30
E. -10
F. 30

$(3, 4, 7) \cdot (-3, -7, 1) = -9 - 28 + 7$

$= -30$

5. If A is a 2×2 invertible matrix such that $(3A)^{-1} = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix}$, what is the (1,1)-entry of A ?

A. -25/3
B. 5/3
C. 5/21
D. 15/7
E. -1/21
F. -5/21

$$(3A)^{-1} = \frac{1}{3}A^{-1} = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & -9 \\ 12 & 15 \end{bmatrix}$$

$$\therefore A = (A^{-1})^{-1} = \begin{bmatrix} -3 & -9 \\ 12 & 15 \end{bmatrix}^{-1} = \frac{1}{63}\begin{bmatrix} 15 & 9 \\ -12 & -3 \end{bmatrix}$$

$$\det A^{-1} = -45 + 108 = 63$$

$\therefore$ (1,1) entry of A is $\frac{15}{63} = \frac{5}{21}$

6. (a) Find all $(a, b, c) \in \mathbf{R}^3$ such that

$$(\star) \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Let W be the set of all $(a, b, c) \in \mathbf{R}^3$ satisfying $(\star)$.

i) Show that W is a subspace of $\mathbf{R}^3$.

ii) Give a basis of W and hence find $\dim W$.

iii) Give a complete geometric description of W.

(a) $LHS = \begin{bmatrix} 1 & 2+a & 3+4a+b \\ 0 & 1 & 4+c \\ 0 & 0 & 1 \end{bmatrix}$ RHS $= \begin{bmatrix} 1 & a+2 & b+2c+3 \\ 0 & 1 & c+4 \\ 0 & 0 & 1 \end{bmatrix}$ (1)

These are equal iff $3+4a+b = b+2c+3$ ie.

$4a - 2c = 0$ or $\underline{2a - c = 0, b \in \mathbb{R}}$ (1)

ie. the 2 are equal iff $(a,b,c) = (a,b,2a)$.

From (a),

(b)i $W = \{(a,b,2a) \mid a,b \in \mathbb{R}\} = \underline{\text{span}\{\overset{v_1}{(1,0,2)}, \overset{v_2}{(0,1,0)}\}}$. $\frac{1}{2}$

These 2 vectors (v_1 & v_2) are clearly l.i., since they aren't ∥.

(i) W is a s.s of $\mathbb{R}^3$ because it's a span. (1)

(ii) $\underline{\{(1,0,2), (0,1,0)\}}$ is a basis (1) of W, so $\dim W = 2$ $\frac{1}{2}$

(iii) W is the plane through 0 with normal $(2,0,-1)$.
(since $(a,b,c) \in W \Leftrightarrow 2a - c = 0$). (1)

7. (a) If A is an $n \times n$ matrix, give 4 different statements which are equivalent to

$$\text{If } x \in \mathbf{R}^n, \text{ then } Ax = 0 \text{ implies } x = 0.$$

(b) Suppose $AB = 0$, where A and B are $n \times n$ matrices. Show that

i) It is impossible for both A and B to be invertible.

ii) $(BA)^2 = 0$.

(a) See your notes $4 @ \left(\frac{1}{2}\right) = 2$

2 (b)(i) Suppose A^{-1} and B^{-1} existed. Then $AB = 0 \Rightarrow A^{-1}(AB) = A^{-1}0 = 0 \Rightarrow B = 0 \Rightarrow B^{-1}B = B^{-1}0 \Rightarrow I = 0$, a contradiction.

2 (ii) $(BA)^2 = (BA)(BA) = B(AB)A = B.0.A = 0$.