

1. Find all p and q for which the following system has infinitely many solutions.

$$\begin{aligned} 2x - y &= 3 \\ x + py &= q \end{aligned}$$

$$[A|b] = \left[\begin{array}{cc|c} 2 & -1 & 3 \\ 1 & p & q \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & p & q \\ 2 & -1 & 3 \end{array} \right]$$

- A. $p = 2, q = -3/2$.
 B. $p = -1/2, q = -3$.
 C. $p = 1/2, q = 3$.
 D. $p = 1/2, q = -3$.
 E. $p = -2, q = 3/2$.
 (F) $p = -1/2, q = 3/2$.

$$\sim \left[\begin{array}{cc|c} 1 & p & q \\ 0 & -1-2p & 3-2q \end{array} \right] \text{ has only}$$

many solns $\Leftrightarrow \text{rank}(A|b) = \text{rank } A < 2$

$$\therefore \left. \begin{aligned} 1+2p &= 0 \\ \text{and } 3-2q &= 0 \end{aligned} \right\} p = -1/2 \text{ \& } q = 3/2$$

2. The volume of a parallelepiped generated by $(1, 1, 0)$, $(1, 0, -1)$ and $(1, 1, 1)$ is:

- A. $\sqrt{3}$
 B. 3
 C. 2
 D. -2
 E. -1
 (F) 1

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} = -1$$

$$\therefore \text{vol} = |-1| = 1$$

3. $S = \{A \in M_{22} \mid A = A^t\}$ is the subspace of symmetric matrices. The dimension of S is:

- A. 5
B. 2
C. 4
D. 3
E. 0
F. 1

$$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

and these 3 matrices are l.i. $\therefore$

$$\dim S = 3.$$

4. Find the eigenvalues of $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} = A$.

- A. -1, 0, 1
B. 1, 2
C. -1, 0
D. 1, 2, 3
E. 1, 3
F. 2, 3

$$|A - xI| = \begin{vmatrix} 5-x & -6 & -6 \\ -1 & 4-x & 2 \\ 3 & -6 & -4-x \end{vmatrix}$$

$$= \begin{vmatrix} 5-x & -6 & 0 \\ -1 & 4-x & -2+x \\ 3 & -6 & 2-x \end{vmatrix} = (2-x) \begin{vmatrix} 5-x & -6 & 0 \\ -1 & 4-x & -1 \\ 3 & -6 & 1 \end{vmatrix}$$

$$= (2-x) \begin{vmatrix} 5-x & -6 & 0 \\ -1 & 4-x & -1 \\ 2 & -2-x & 0 \end{vmatrix} = (2-x) \begin{vmatrix} 5-x & -6 \\ 2 & -2-x \end{vmatrix} = (2-x) \{x^2 - 3x - 10 + 12\}$$

$$= (2-x)(x-1)(x-2)$$

$$= 0$$

$$\Leftrightarrow x=1 \text{ or } x=2.$$

Please record your answers on the title page.

5. Compute $\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}^{1000001}$

A. $\begin{bmatrix} -1000001 & -3000003 \\ 0 & 1000001 \end{bmatrix}$

B. $\begin{bmatrix} -1 & 3^{1000001} \\ 0 & 1 \end{bmatrix}$

☒ C. $\begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1000001 & -3000003 \\ 0 & -1000001 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$

If $A = \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}$,

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\therefore A^{2n+1} = A$$

$$\therefore A^{1000001} = A$$

6. B is a 4×4 matrix with $\det(2BB^t) = 64$. Find $|\det(3B^2B^t)|$.

A. 192

B. 256

C. 324

☒ D. 648

E. 128

F. 640

$$64 = \det(2BB^t) = 2^4 \cdot \det^2 B \Rightarrow \det B = \pm \sqrt{4} = \pm 2$$

$$\therefore |\det(3B^2B^t)| = 3^4 |\det^3 B| = 3^4 \cdot 2^3 = 81 \cdot 8 = 648$$

7. A Norwegian Blue parrot is advised by a nutritionist to take 14 units of vitamin A, 15 units of vitamin D and 29 units of vitamin E each day. The parrot can choose from three brands I, II and III, and the amount of each vitamin in every capsule of the various brands is given below:

	I	II	III
vitamin A	2	1	1
vitamin D	3	3	0
vitamin E	5	4	1

- a) Write down a system of equations and constraints that determine the possible combinations of numbers of capsules of each brand that will provide exactly the required amounts of vitamins for the parrot. Don't forget to define your variables.
- b) Solve the constrained linear system of (a).
- c) If the respective costs per capsule of brands I, II and III are 30, 15 and 12, determine the choice which will minimize the total cost.

a) Let $x =$ # capsules of brand I the NBP takes
 $y =$ " " " " "
 $z =$ " " " " "

Then $\begin{cases} x, y, z \geq 0 \\ \text{and } x, y, z \in \mathbb{Z} \end{cases}$

A: $2x + y + z = 14$
D: $3x + 3y + 0z = 15$
E: $5x + 4y + z = 29$

b) $\begin{bmatrix} 2 & 1 & 1 & | & 14 \\ 3 & 3 & 0 & | & 15 \\ 5 & 4 & 1 & | & 29 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 1 & | & 9 \\ 0 & 1 & -1 & | & -4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

$\therefore \begin{cases} x = 9 - z \\ y = -4 + z \\ z \geq 0 \end{cases}$

$x, y, z \in \mathbb{Z} \Leftrightarrow z \in \mathbb{Z}$
 $x \geq 0 \Leftrightarrow z \leq 9$
 $y \geq 0 \Leftrightarrow z \geq 4$
 $z \geq 0 \Leftrightarrow \text{no new info.}$

$\therefore$

z	4	5	6	7	8	9
x	5	4	3	2	1	0
y	0	1	2	3	4	5
z	4	5	6	7	8	9

are the possibilities.

c) Cost = $30x + 15y + 12z = 30(9 - z) + 15(-4 + z) + 12z = 210 - 3z$

This is minimized when z is maximized i.e. $z = 9$
 $(x, y, z) = (0, 5, 9)$
the least expensive choice

8. Let $S = \left\{ \overset{A_1}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{A_2}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}, \overset{A_3}{\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}} \right\}$.

a) Show that S is linearly dependent.

b) Explain why $\text{span } S = \text{span} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

c) Find the dimension of $\text{span } S$.

d) Is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \text{span } S$?

a) Either: note that $A_3 = A_1 + 2A_2$ (1/2)

OR, solve: $x A_1 + y A_2 + z A_3 = 0 \Leftrightarrow \begin{bmatrix} x+z & x-y-z \\ y+z & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ OR

$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right]$; this has only many soln, so (1/2)

$\{A_1, A_2, A_3\}$ is l.d.

(1b) Since $A_3 \in \text{span} \{A_1, A_2\}$, $\text{span} \{A_1, A_2, A_3\} = \text{span} \{A_1, A_2\}$

c) $\dim \text{span } S = \dim \text{span} \{A_1, A_2\}$. $x A_1 + y A_2 = 0 \Leftrightarrow \begin{pmatrix} x & x-y \\ y & 0 \end{pmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow x=y=0$ (1) $\therefore \{A_1, A_2\}$ is l.i. & hence a basis of $\text{span } S$.

$\therefore \dim \text{span } S = 2$. (1)

d) (1/2) $x A_1 + y A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{array} \right]$, which is (1/2)

inconsistent.

$\therefore \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \notin \text{span} \{A_1, A_2\} = \text{span } S$.

1/2

9. Consider the 2 planes $\{(x, y, z) \mid x - 3y + 4z = 0\}$ and $\{(x, y, z) \mid 2x - 7y + 9z = 0\}$.

- If V is the intersection of these planes, give a geometric description of V , and explain why V is a subspace of $\mathbb{R}^3$. (You do not need to use the subspace test here.)
- Find a basis of V .
- If $v = n_1 \times n_2$ is the cross product of the normals n_1 and n_2 of the planes above, show that $V = \text{span}\{v\}$.
- Show that $\{n_1, n_2, v\}$ is a basis of $\mathbb{R}^3$. Is this an orthogonal basis?

a) These planes are not $\parallel$, so V is a line passing through the origin with direction given by $n_1 \times n_2$ where n_1, n_2 are the normals of these planes. It is a subspace of $\mathbb{R}^3$ because it is a line passing through the origin.

b)
$$\begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 2 & -7 & 9 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix} \quad \begin{matrix} x = -s \\ y = s \\ z = s \end{matrix} \quad \therefore V = \{(-s, s, s) \mid s \in \mathbb{R}\} = \text{span}\{(-1, 1, 1)\}.$$

$\{(-1, 1, 1)\}$ is a basis for V .

c)
$$v = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & -3 & 4 \\ 2 & -7 & 9 \end{vmatrix} = (1, -1, -1)$$

But

$$\text{span}\{(1, -1, -1)\} = \text{span}\{(-1, 1, 1)\} = V.$$

d) $\{n_1, n_2, v\}$ is a basis of $\mathbb{R}^3$ because $\det[n_1, n_2, v] \neq 0$.

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & -7 & 9 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 5 \\ 2 & -5 & 11 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ -5 & 11 \end{vmatrix} = 3 \neq 0$$
 $\therefore \{n_1, n_2, v\}$ is a basis of $\mathbb{R}^3$.

Since $n_1 \cdot n_2 = 2 + 21 + 36 \neq 0$, $\{n_1, n_2, v\}$ is not an orthogonal basis of $\mathbb{R}^3$.

10. Let $P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

- a) Show that $P^2 = P$.
- b) Find a basis for the column space $\text{col}(P)$ of P , i.e. the subspace of $\mathbb{R}^3$ spanned by the columns of P . Hence, determine $\dim \text{col}(P)$ and compare it with $\text{rank } P$.
- c) Explain why $\det P = 0$ without computing $\det P$. (Use a theorem!)
- d) If Q is any 3 by 3 matrix satisfying $Q^2 = Q$, show that 0 or 1 are the only possible eigenvalues of Q .

a) $P^2 = \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = P$

b) $\text{Col}(P) = \text{span} \{ (1, 1, 1), (1, 1, 1), (1, 1, 1) \} = \text{span} \{ (1, 1, 1) \}$
 Since $(1, 1, 1) \neq 0$, $\{ (1, 1, 1) \}$ is a basis of $\text{col}(P)$

$\therefore \dim \text{col } P = 1$

$\text{rank } P = \text{rank} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 1$
 $\therefore \text{rank } P = \dim \text{col}(P)$

c) $\text{rank } P = 1 < 3$
 P is not invertible. $\det P = 0$

(or: P has 2 rows (cols) which are the same, $\therefore \det P = 0$).

d) Suppose $Qv = \lambda v$, $v \neq 0$. Then $Q^2v = Q(\lambda v) = \lambda^2 v = Qv = \lambda v$

So $(\lambda^2 - \lambda)v = 0$. Since $v \neq 0$, $\lambda^2 - \lambda = \lambda(1 - \lambda) = 0$
 $\therefore \lambda = 0$ or $\lambda = 1$.

11. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 2 & 2 & 0 \end{bmatrix}$. The eigenvalues of A are -2 and 4.

(1) a) Find a basis of $E_{-2} = \{x \in \mathbb{R}^3 \mid (A + 2I)x = 0\}$.

(1) b) Find a basis of $E_4 = \{x \in \mathbb{R}^3 \mid (A - 4I)x = 0\}$.

(1) c) Show that the set consisting of all vectors in the bases for E_{-2} and E_4 is a basis for $\mathbb{R}^3$.

(2) d) If possible, find an invertible matrix P such that $P^{-1}AP = D$ is diagonal and give this diagonal matrix D .

$$a) [A + 2I | 0] = \begin{bmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -2 & 0 \\ 0 & 24 & 12 & 0 \\ 0 & 12 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} x = -\frac{1}{2}z \\ y = -\frac{1}{2}z \\ z = 0 \end{matrix}$$

$\therefore$ f.s. is $(-\frac{1}{2}, -\frac{1}{2}, 1)$, $\therefore \{(-\frac{1}{2}, -\frac{1}{2}, 1)\}$ is a basis for E_{-2}
(or $\{(1, 1, -2)\}$) (1)

$$b) [A - 4I | 0] = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} x = -1 + 2t \\ y = t \\ z = t \end{matrix}$$

$\therefore$ f.s. are $\{(-1, 1, 0), (2, 0, 1)\}$, which is a basis.
(1/2)

c) $\{(1, 1, -2), (-1, 1, 0), (2, 0, 1)\}$ is a basis of $\mathbb{R}^3$ since. (1)

$$|P| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 2 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{vmatrix} = - \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = 6 \neq 0. \text{ (Hence, } P = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ is invertible)} \quad (1/2)$$

d) Let (1) P be the invertible matrix as in c). Then since the cols of P are eigenvectors of A in order -2, 4, 4, we have $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$ (1)

12. State whether the following are always true or could be false. If true, explain why, if false, give a numerical example to illustrate.

- a) If a 3 by 5 matrix A has rank 3, the system $Ax = 0$ has a unique solution.
- b) If $\{u, v, w\}$ are 3 vectors in a vector space V , then $\{u, v, w\}$ is linearly independent if and only if $au + bv + cw = 0$ where the scalars a, b and c satisfy $a = b = c = 0$.
- c) If two 3 by 3 matrices A and B satisfy $AB = -BA$, then at least one of A or B is not invertible.

a) This is false. e.g. $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$ gen'l soln here to $Ax = 0$
 has 2 parameters, so $Ax = 0$ has only many solns.
 (1)

b) This is false. e.g. let $u = e_1, v = e_1, w = e_1$. Then
 $a e_1 + b e_1 + c e_1 = 0$ where $a = b = c = 0$. But $\{e_1, e_1, e_1\}$
 is not l.i. ($e_1 - e_1 = 0$, e.g.)
 (1)

c) This is true: $AB = -BA \Rightarrow \det A \det B = (-1)^3 \det B \det A$
 $\Rightarrow \det A \det B = -\det A \det B$ or $2 \det A \det B = 0$. $\therefore \det A \det B = 0$
 $\therefore$ at least one of $\det A$ or $\det B = 0$. If $\det A = 0$, A is not invertible; similarly for B .
 (1)