Mat 1341A Mid-Term 1999 Solutions

1. If the vector $v = (2, -1, 0)$ is written as $v = c_1v_1 + c_2v_2 + c_3v_3$, where $\{v_1, v_2, v_3\}$ is the orthonormal basis with

$$v_1 = \frac{\sqrt{2}}{2}(1,0,1), \quad v_2 = (0,1,0), \quad \text{and } v_3 = \frac{\sqrt{2}}{2}(1,0,-1),$$

then (c_1, c_2, c_3) is

A. $(-1, \sqrt{2}, \sqrt{2})$
B. $(\sqrt{2}, -1, \sqrt{2})$
C. $(-\sqrt{2}, 1, \sqrt{2})$
D. $(1, -\sqrt{2}, \sqrt{2})$
E. $(\sqrt{2}, 1, \sqrt{2})$
F. $(1, -\sqrt{2}, -\sqrt{2})$

$$c_1 = \frac{v \cdot v_1}{\|v_1\|^2} = \frac{(2,-1,0) \cdot \frac{\sqrt{2}}{2}(1,0,1)}{1} = \sqrt{2}$$

$$c_2 = \frac{v \cdot v_2}{\|v_2\|^2} = \frac{(2,-1,0) \cdot (0,1,0)}{1} = -1$$

$$c_3 = \frac{v \cdot v_3}{\|v_3\|^2} = \frac{(2,-1,0) \cdot \frac{\sqrt{2}}{2}(1,0,-1)}{1} = \sqrt{2}$$

Please record your answer on the title page.

2. Each statement below is True or False.

- Every system of 2 equations in 2 unknowns has a unique solution. False : e.g. $x + y = 0$, $2x + 2y = 0$
- The set of solutions of the system consisting of the single equation

$$2x - 3y = 0$$

in the three variables x, y and z is a subspace of $\mathbf{R}^3$. True : this is a plane through 0.

- There is a linear system in 2 variables which is inconsistent. True : e.g. $x + y = 0$, $x + y = 1$

Choose the correct sequence from the possibilities below.

A. True, True, False.
B. True, False, True.
C. True, False, False.
D. False, True, True.
E. False, False, True.
F. False, True, False.

3. Let $v_0 = (1,0,1)$ and define

$$U = \{(x,y,z) \in \mathbf{R}^3 \mid x+z=0\}.$$

i) Show that $v=(x,y,z)$ belongs to U if and only if v is orthogonal to v_0.

ii) Show that $u_1 = (1,0,0) - \mathrm{proj}_{v_0}(1,0,0)$ and $u_2 = (0,1,0) - \mathrm{proj}_{v_0}(0,1,0)$ are orthogonal and belong to U.

iii) Give a geometric description of U and show that $\{u_1, u_2\}$ is a basis of U.

iv) If $W = \mathrm{span}\{u_1, u_2, v_0\}$, what is $\dim W$? Is $W = \mathbf{R}^3$?

i) $(x,y,z) \in U \iff x+z=0 \iff (x,y,z)\cdot(1,0,1)=0 \iff v=(x,y,z)\cdot v_0 = 0$ (1/2)
i.e. v is orthogonal to v_0.

ii) $u_1 = (1,0,0) - \left\{\frac{(1,0,0)\cdot(1,0,1)}{2}\right\}(1,0,1) = (\frac{1}{2},0,-\frac{1}{2})$ (1/2)

$u_2 = (0,1,0) - \left\{\frac{(0,1,0)\cdot(1,0,1)}{2}\right\}(1,0,1) = (0,1,0)$ (1/2)

Then, $u_1 \cdot u_2 = 0$, so u_1 & u_2 are orthogonal. (1/2) Moreover,
$u_1 \cdot v_0 = 0 = u_2 \cdot v_0$, so u_1 & u_2 belong to U. (1/2)

iii) U is the plane through O in $\mathbb{R}^3$ with normal $(1,0,1)$. (1) (so $\dim U = 2$)
Because $\{u_1, u_2\}$ is an orthogonal set, it is l.i. (1/2) Moreover, $\{u_1,u_2\} \subset U$.
Since every l.i. set of 2 vectors in a space of dimension 2 is a basis,
$\{u_1, u_2\}$ is a basis of U. (1/2)

iv) Previous calculations show that $\{u_1, u_2, v_0\}$ is an orthogonal set,
and so it is l.i. (1/2) Hence, $\{u_1,u_2,v_0\}$ is a basis of W, so $\dim W = 3$. (1/2)

Yes, $W = \mathbb{R}^3$ because, being a l.i. set with 3 vectors in a space $(\mathbb{R}^3)$
of dimension 3, $\{u_1,u_2,v_0\}$ is also a basis of $\mathbb{R}^3$. i.e. (1/2)
$W = \mathrm{span}\{u_1,u_2,v_0\} = \mathbb{R}^3$.

4.

Consider the vector space $\mathbf{F}[1,2] = \{f \mid f \text{ is a real-valued function defined on } [1,2]\}$ and recall that the zero of $\mathbf{F}[1,2]$ is the function that has the value 0 for all $x \in [1,2]$.

Suppose $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ and let $W = \text{span}\{f, g\}$.

3 i) Show that $\{f, g\}$ is linearly independent. What is $\dim W$?

1 ii) If $h(x) = \frac{2x-3}{x^2}$, show that $h \in W$.

2 iii) What is the dimension of $\text{span}\{f, g, h\}$?

i) Suppose $a\left(\frac{1}{x}\right) + b\left(\frac{1}{x^2}\right) = 0$ for all $x \in [1,2]$. 1

At $x = 1$: $a + b = 0$

$x = 2$: $\frac{a}{2} + \frac{b}{4} = 0 \Rightarrow 2a + b = 0$ } $a = 0$, so $b = 0$ 1

Thus $\{f, g\}$ is l.i., so $\dim W = 2$ 1 ($\{f, g\}$ is a basis for W)

ii) $h(x) = \frac{2x-3}{x^2} = \frac{2}{x} - \frac{3}{x^2}$ so $h = 2f - 3g \in \text{span}\{f, g\} = W$.

1

iii) Since $h \in \text{span}\{f, g\}$, $\text{span}\{f, g, h\} = \text{span}\{f, g\} = W$. 1

Thus, $\dim \text{span}\{f, g, h\} = \dim W = 2$. 1

5. Suppose $\mathcal{B} = \{u, v, w\}$ is a basis of a vector space V.

2.5 i) State the two properties of $\mathcal{B}$ that make it a basis of V. What is $\dim V$?

1.5 ii) Show that $\mathcal{C} = \{u + 2v, u + 3w, v + w\}$ is linearly independent.

2 iii) Is $\mathcal{C}$ a basis of V? You must justify your answer, as usual.

i). $\mathcal{B}$ is l.i., and spans V. $\dim V = 3$. (3 vectors in the basis)

1 1 ½

ii) Suppose $a(u + 2v) + b(u + 3w) + c(v + w) = 0$ ½

Then, $(a+b)u + (2a+c)v + (3b+c)w = 0$.

Since $\mathcal{B}$ is l.i., we must have ½

$$\left.\begin{array}{l} a + b = 0 \\ \left.\begin{array}{l} 2a + c = 0 \\ 3b + c = 0 \end{array}\right\} 2a - 3b = 0 \end{array}\right\} a = b = 0$$

$\Rightarrow a = b = c = 0$. ½

Hence $\mathcal{C}$ is l.i.

iii) Yes, 1 because 1 $\mathcal{C}$ is a l.i. subset of V which contains $3 = \dim V$ vectors.