134A.F99 mini-test 1 solutions

1. Which of $U = \{ (x, x+y, xy) \mid x, y \in \mathbf{R} \}$, $V = \{ (y, x, 2x+y) \mid x, y \in \mathbf{R} \}$ and $W = \{ (x, y, x^2+y^2) \mid x, y \in \mathbf{R} \}$ are subspaces of $\mathbf{R}^3$?

A. U and W only
B. U and V only
C. W only
D. U only
E. V and W only
F. V only

U: $(1,2,1) \in U$ but $2.(1,2,1) = (2,4,2) \notin U$, so U is not a s.s.

V: $(y, x, 2x+y) = y(1,0,1) + x(0,1,2)$
$\therefore V = \text{span}\{(1,0,1), (0,1,2)\}$, so V is a s.s.

W: $(1,0,1) \in W$ but $2.(1,0,1) = (2,0,2) \notin W$
$\therefore W$ is not a s.s.

2. Which of the following is a subspace of $\mathbf{R}^3$?

A. $\{ (a, b, c) \mid a + b + c = 1 \}$: doesn't contain $(0,0,0)$ $\therefore$ not a s.s.
B. None of these
C. $\{ (a, b, c) \mid a = 1, b = 0 \text{ and } c = 1 \}$: doesn't contain $(0,0,0)$ $\therefore$ not a s.s.
D. $\{ (a, b, c) \mid a > 0 \text{ and } b > 0 \}$: doesn't contain $(0,0,0)$ $\therefore$ not a s.s.
E. $\{ (a, b, c) \mid a = b = 0 \} = \text{span}\{(0,0,1)\}$ $\therefore$ is a s.s.
F. $\{ (a, b, c) \mid a > 0 \text{ and } b < 0 \}$: doesn't contain $(0,0,0)$ $\therefore$ not a s.s.

3. If $\mathbf{F(R)} = \{ f \mid f: \mathbf{R} \to \mathbf{R}\}$, find all real constants a so that

$W = \{ f \in \mathbf{F(R)} \mid f(1) = f(a) = a \}$ is a subspace of $\mathbf{F(R)}$.

A. $a = 1$
B. Any $a \in \mathbf{R}$
C. $a = 0$
D. $a = -1$
E. $a > 0$
F. $a < 0$

If W is a subspace, the zero function $Z(x) = 0$, (for all $x \in \mathbb{R}$) must be in W. Then, $0 = Z(1) = Z(a) = a$ implies $a = 0$. (Hence C. is the only possibility) It's easy to check that if $a = 0$, then W is a s.s. of $F(R)$.

4. Which two of the following are <u>not</u> subspaces of $\mathbf{R}^4$?

$R = \{(a, b, c, d) \mid c = a + 2b, d = a - 3b\}$ = span$\{(1,0,1,1), (0,1,2,-3)\}$ is a s.s.
$S = \{(a, b, c, d) \mid a = 0, b = 0\}$ = Span$\{(0,0,1,0), (0,0,0,1)\}$ $\therefore$ is a s.s.
$T = \{(a, b, c, d) \mid a - b = 2, c = d\}$ doesn't contain $(0,0,0,0)$ $\therefore$ not a s.s.
$U = \{(a, b, c, d) \mid a \geq 0, b \geq 0\}$ $(1,1,0,0) \in U$ but $-(1,1,0,0) \notin U$ $\therefore$ not a s.s.

A. S and U
B. R and S
C. S and T
D. T and U
E. R and T
F. R and U

5. Find the volume of the parallelepiped determined by the vectors (1, 1, -1), (2, 0, 1) and (1, -1, 3).

Ⓐ 2
B. 8
C. 4
D. –2
E. –8
F. 16

$$\text{vol} = |(1,1,-1)\cdot(2,0,1)\times(1,-1,3)|.$$

$$(1,1,-1)\cdot(2,0,1)\times(1,-1,3)$$

$$= (1,1,-1)\cdot\begin{vmatrix} i & j & k \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= (1,1,-1)\cdot(1,-5,-2)$$

$$= 1-5+2 = -2$$

$$\therefore \text{vol} = |-2| = 2$$

4. Let $v_0 = (0,1,0)$ and define a subset U of $\mathbf{R}^3$ by

$$U = \{u \in \mathbf{R}^3 \mid u \times v_0 = 0\}.$$

i) Show that if $v \in \text{span}\{v_0\}$ then $v \in U$.

ii) Show that if $v \in U$, then $v \in \text{span}\{v_0\}$.

iii) Give a geometric description of U.

iv) Is U a subspace of $\mathbf{R}^3$? (You should not need to use the subspace test here).

i) If $v \in \text{span}\{v_0\}$, then $v = kv_0$ for some $k \in \mathbb{R}$. Then
$v \times v_0 = (kv_0) \times v_0 = k(v_0 \times v_0) = k.0 = 0$, so $v \in \mathcal{U}$.
Hence $\text{span}\{v_0\} \subseteq \mathcal{U}$.

ii) Suppose $v = (x, y, z) \in \mathcal{U}$. Then
$$(0,0,0) = \begin{vmatrix} i & j & k \\ x & y & z \\ 0 & 1 & 0 \end{vmatrix} = (-z, 0, x),$$
so $x = z = 0$. Hence $v = (0, y, 0) = y \cdot (0,1,0)$
$= y \cdot v_0 \in \text{span}\{v_0\}$

(Hence $\mathcal{U} \subseteq \text{span}\{v_0\}$ and $\mathcal{U} \subseteq \text{span}\{v_0\}$, so
$\mathcal{U} = \text{span}\{v_0\}$)

iii) i) & ii) show that $\mathcal{U}$ is the line through O in R^3 with direction $v_0 = (0,1,0)$.

iv) Yes, because all lines through O in R^3 are subspaces of $\mathbb{R}^3$. (or: Yes, because $\mathcal{U} = \text{span}\{v_0\}$, and every span is a subspace.)

5. Let $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$ and consider the subset

$$W = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-x) = f(x), \text{ for all } x \in \mathbf{R}\}.$$

i) Use the subspace test to show that W is a subspace of $\mathbf{F}(\mathbf{R})$.

ii) Show that if $c(x) = \sin^2 x$, then $c \in W$. (Hint: recall that $x \mapsto \sin x$ is an odd function.)

iii) If $g(x) = 1$ for all $x \in \mathbf{R}$, and c is defined as in (ii), show that $\operatorname{span}\{c, g\} \subset W$.

i) 1) Let $Z(x) = 0$ for all $x \in \mathbb{R}$ be the zero function. Then, for any $x \in \mathbb{R}$, $Z(-x) = 0 =$ So $Z \in W$

2) If $f, g \in W$, then, for any $x \in \mathbb{R}$, $(f+g)(-x) = f(-x) + g(-x) = f(x) - g(x)$ $= (f(x) + g(x)) = (f+g)(x)$ $\therefore f+g \in W$. So W is closed under addition.

3) If $f \in W$ and $k \in \mathbb{R}$, then, for any $x \in \mathbb{R}$, $(kf)(-x) = k.f(-x) = k(f(x))$ $= k.f(x) = (kf)(x)$ $\therefore kf \in W$. So W is closed under multiplication by scalars.

By the S.S. test, W is a subspace test of $F(R)$.

ii) For any $x \in \mathbb{R}$, $c(-x) = \sin^2(-x) = [\sin(-x)]^2 = [-\sin x]^2 = \sin^2 x = c(x)$ holds.
$\therefore c \in W$.

iii) Since, for any $x \in \mathbb{R}$, $g(-x) = 1 = g(x)$, $g \in W$.
If $f \in \operatorname{span}\{c, g\}$, then $f = k_1 c + k_2 g$ for $k_1, k_2 \in \mathbb{R}$.
Hence, for any $x \in \mathbb{R}$, $f(-x) = k_1 c(-x) + k_2 g(-x) = k_1 c(x) + k_2 g(x) = f(x)$. Hence, $f \in W$. Thus, $\operatorname{span}\{c, g\} \subseteq W$.