1. Which of the following are bases for R^3 ? $\dim \mathbb{R}^3 = 3 \therefore 3$ vectors in a basis

(1) { (4, 2, 0), (1, 4, 1), (1, –10, -3) }
(2) { (-1, 2, 3), (3, 3, 2) } impossible
(3) { (-1, 3, -5), (1, -2, 4), (2, 0, 4), (5, 1, 9) } impossible

A. (2) only
B. (2) and (3)
C. All three
D. (1) and (2)
E. None of them
F. (1) only

(1) is a basis $\Leftrightarrow$ it is l.i.

$$\Leftrightarrow \text{rank}\begin{bmatrix} 4 & 1 & 1 \\ 2 & 4 & -10 \\ 0 & 1 & -3 \end{bmatrix} = 3.$$

$$\begin{bmatrix} 4 & 1 & 1 \\ 2 & 4 & -10 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -3 \\ 0 & -7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \text{rank} = 2,$$

so three vectors in (1) are l.d. $\therefore$ not a basis.

2. If $A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 3 & -5 & 7 & 8 \end{bmatrix}$, a basis for the solution space of $Ax = 0$ is

A. {(1, 2, 1, 0)}
B. {(1, 2, 1, 0), (4, 4, 0, 1)}
C. {(4, 4, 0, 1)}
D. {(1, 0), (0, 1)}
E. {(0, 1, -2, -4)}
F. {(1, 0, -1, -4), (0, 1, -2, -4)}

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 3 & -5 & 7 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & 1 & -2 & -4 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \sim \end{array} \begin{array}{c} \quad\quad\quad s \quad\ t \\ \left[\begin{array}{cccc|c} 1 & 0 & -1 & -4 & 0 \\ 0 & 1 & -2 & -4 & 0 \end{array}\right] \end{array} \quad \therefore \begin{aligned} x &= s + 4t \\ y &= 2s + 4t \\ z &= s \\ w &= t \end{aligned}$$

$\therefore \{(1, 2, 1, 0), (4, 4, 0, 1)\}$ is a basis (of fundamental soln.)

3. Find <u>all</u> t such that

(1, t, 3, 4) $\in$ span {(1, -2, 3, 4), (-3, 6, -5, -16), (-1, 2, -5, -2)}.

A. t = -1
B. t = 0
C. t $\neq$ -2
D. t $\neq$ -1
E. t = -2
F. t $\neq$ 0

We solve $\left[\begin{array}{ccc|c} 1 & -3 & -1 & 1 \\ -2 & 6 & 2 & t \\ 3 & -5 & -5 & 3 \\ 4 & -16 & -2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & 1 \\ 0 & 0 & 0 & t+2 \\ 0 & 4 & -2 & \\ 0 & -4 & 2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & 1 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & t+2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which is consistent $\Leftrightarrow t=-2$.

4. The inverse of the matrix $\begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix}$ $\left\{ \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix}^{-1} = \frac{1}{1+x^2} \begin{bmatrix} 1 & -x \\ x & 1 \end{bmatrix} \right.$

A. is $\frac{1}{1+x^2}\begin{bmatrix} 1 & -x \\ x & 1 \end{bmatrix}$

B. is $\frac{1}{1+x^2}\begin{bmatrix} x & 1 \\ 1 & -x \end{bmatrix}$

C. is $\frac{1}{1-x^2}\begin{bmatrix} x & 1 \\ 1 & -x \end{bmatrix}$

D. is $\frac{1}{1-x^2}\begin{bmatrix} 1 & -x \\ x & 1 \end{bmatrix}$

E. exists if and only if $x \neq 0$.

F. is $\frac{1}{1+x^2}\begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix}$

5. If two n×n matrices satisfy $A^t = B$ and $B^t = -B$, then $(ABA)^t$ is always equal to:

A. B^3
B. B^tB
C. B^tBB^t
D. $-B^3$
E. B^tB^tB
F. $(ABA)^t$ is not defined

$$(ABA)^t = A^t B^t A^t$$
$$= B(-B)B$$
$$= -B^3$$

6. Consider the linear system

$$\begin{array}{rcrcrcr} x & & & + & z & = & -2 \\ x & + & y & & & = & 0 \\ 3x & + & y & + & cz & = & d \end{array}$$

a) If $[A|b]$ is the augmented matrix of the system above, find rank A and rank $[A|b]$ for all values of c and d.

b) Find all c and d so that this system has
  i) a unique solution,
  ii) infinitely many solutions, and
  iii) no solutions.

c) In case (ii) above, give a geometric description of the set of solutions.

a) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & c & d \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & c-3 & d+6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & c-2 & d+4 \end{array}\right]$

$\therefore$ rank $A = \begin{cases} 3 & \text{if } c \neq 2 \\ 2 & \text{if } c = 2 \end{cases}$ & rank $[A|b] = \begin{cases} 3 & \text{if } c \neq 2 \\ 2 & \text{if } c = 2 \text{ \& } d = -4 \\ 3 & \text{if } c = 2 \text{ \& } d \neq -4 \end{cases}$

b) i) unique solⁿ $\Leftrightarrow$ rank $A =$ rank $[A|b] = 3 \Leftrightarrow c \neq 2$

ii) ∞ly many solⁿ $\Leftrightarrow$ rank $A =$ rank $[A|b] < 3 \Leftrightarrow c = 2,\ d = -4$

iii) inconsistent $\Leftrightarrow$ rank $A <$ rank $[A|b] \Leftrightarrow c = 2$ & $d \neq -4$

c) If $c = 2$ & $d = -4$, $[A|b] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\therefore$ $\begin{array}{l} x = -2 - s \\ y = 2 + s \\ z = s \end{array}$ $\quad s \in \mathbb{R}$

$\therefore$ set of solutions $= \{(-2-s, 2+s, s) \mid s \in \mathbb{R}\}$

$= \{(-2, 2, 0) + s(-1, 1, 1) \mid s \in \mathbb{R}\}$

which is a line in $\mathbb{R}^3$ through $(-2,2,0)$ with direction $(-1,1,1)$.

2.5

7. a) An amusement park charges \$5 for adults, \$3 for youths and \$1 for children. If 103 people enter and pay a total of \$111, write down a system of equations, together with all constraints on your variables, which will describe the possible numbers of adults, youths and children who visited the park.

Don't forget to define your variables.

DO NOT, REPEAT, DO NOT SOLVE THIS SYSTEM.

3.5 b) Find all solutions (x, y, z) of

$$\begin{array}{rcrcrcr} x & + & y & & & = & 5 \\ 2x & + & y & + & z & = & 13 \\ 4x & + & 3y & + & z & = & 23 \end{array}$$

which satisfy $x \geq 3$, $y \geq 0$ and $z \geq 1$, and such that x, y, z are integers.

a) Let x = # of adults who visited the park,
y = " youths "
& z = " children " $\Big\}$ $\frac{1}{2}$

Then x, y, z are non-negative integers $\Big\}$ 1

$x + y + z = 103$ (103 enter)
$5x + 3y + z = 111$ (total paid) $\Big\}$ 1

b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 2 & 1 & 1 & 13 \\ 4 & 3 & 1 & 23 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} & & s & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ $\frac{1}{2}$

$\frac{1}{2}$ $\therefore \left\{ \begin{array}{l} x = 8 - s \\ y = -3 + s \quad s \in \mathbb{R}. \\ z = s \end{array} \right.$

Constraints:
$x, y, z \in \mathbb{Z} \Leftrightarrow s \in \mathbb{Z}.$ $1/2$

$x \geq 3 \Leftrightarrow 8 - s \geq 3 \Leftrightarrow s \leq 5$
$y \geq 0 \Leftrightarrow -3 + s \geq 0 \Leftrightarrow s \geq 3$
$z \geq 1 \Leftrightarrow s \geq 1$ $\Big\}$ 1

$\therefore s \in \{3, 4, 5\}$

All solutions are: $\{(5, 0, 3), (4, 1, 4), (3, 2, 5)\}$ $\Big\}$ 1