Mat B41A. F02 Mid-term.

1. If the coefficient matrix A in a homogeneous system of 12 equations in 16 unknowns is known to have rank 5, how many parameters are there in the general solution?

A. 10
B. 7
C. 11
D. none
E. 12
F. 16

parameters = # unknowns − rank A

= 16 − 5

= 11

2. Find the (2, 3)-entry of the product

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 5 & 1 \\ 4 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 5 & 1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$$

$(0, 2, 5, 1) \cdot (1, 2, 0, 3) = 7$

A. 4
B. 5
C. 6
D. 7
E. 8
F. 9

3. Let A be an 8×6 matrix such that $Ax = 0$ has only the trivial solution $x = 0$.

- What is the rank of A?
- Do the columns of A span $\mathbf{R}^8$?

A. 0, Yes
B. 6, Yes
C. 6, No
D. 8, Yes
E. 8, No
F. 2, Yes

Unique sol^n $\Leftrightarrow$ consistent (✓ homog.)

& rank$A$ = # col of A

$\therefore$ rank $A = 6$

A has only $\underline{6}$ columns, and the dimension of $\mathbb{R}^8 = 8$. Any spanning set of $\mathbb{R}^8$ has at least $\dim \mathbb{R}^8 = 8$ vectors. So $\underline{\text{NO}}$.

4. Suppose $a, c \in \mathbf{R}$ and consider the linear system in x, y and z:

$$\begin{array}{rcrcrcl} x & & & + & az & = & 1 \\ 2x & + & y & + & 2az & = & 3 \\ 3x & + & y & + & 4az & = & c \end{array}$$

a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank A and rank$[\,A\,|\,b\,]$ for all values of a and c.

b) Using part (a), find all values of a and c so that this system has

(i) a unique solution,

(ii) infinitely many solutions, or

(iii) no solutions.

c) In case b(ii) above, give a geometric description of the set of solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & a & 1 \\ 2 & 1 & 2a & 3 \\ 3 & 1 & 4a & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & a & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & a & c-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & a & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & a & c-4 \end{array}\right]$$

a) $\operatorname{rank} A = \begin{cases} 2 & \text{if } a = 0 \\ 3 & \text{if } a \neq 0 \end{cases}$

$\operatorname{rank}[A|b] = \begin{cases} 2 & \text{if } a = 0 \text{ and } c = 4 \\ 3 & \text{otherwise} \end{cases}$

b) (i) Unique sol^n exists $\Leftrightarrow$ rank A = rank $[A|b]$ = # variables = 3 ; hence $\underline{a \neq 0}$

(ii) ∞^{ly} many sol^{ns} $\Leftrightarrow$ rank $[A|b]$ = rank A < # variables ; $\underline{\text{hence } a = 0 \text{ and } c = 4}$

(iii) no solutions $\Leftrightarrow$ rank $[A|b]$ > rank A ; hence $\underline{a = 0,\ c \neq 4}$

c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x = 1 \\ y = 1 \\ z = s \end{array}$; $s \in \mathbb{R}$. This is the line through $(1,1,0)$ with direction $(0,0,1)$.

5. Let $\mathbf{F}[0,\pi] = \{f \mid f : [0,\pi] \to \mathbf{R}\}$ be the vector space of real-valued functions defined on $[0,\pi]$.

Define three functions in $\mathbf{F}([0,\pi])$ by $f(x) = 1$, $g(x) = \sin 2x$ and $h(x) = \cos x$ and let

$$W = \operatorname{span}\{f, g, h\}. \quad *$$

a) Show that $\{f, g, h\}$ is linearly independent.

b) Use (a) to show that $\{f+g, f+h, g+h\}$ is linearly independent.

c) Give two different bases of W.

d) What is $\dim W$?

a) Suppose $a1 + b\sin 2x + c\cos x = 0$, $\forall x \in [0,\pi]$.

$$\left.\begin{array}{lll} x = 0 & \Rightarrow & a \quad\quad + c = 0 \\ x = \frac{\pi}{4} & \Rightarrow & a + b + \frac{\sqrt{2}}{2}c = 0 \\ x = \frac{\pi}{2} & \Rightarrow & a \quad\quad\quad\quad = 0 \end{array}\right\} \Rightarrow a = b = c = 0$$

$\therefore \{f, g, h\}$ is l.i.

b) Suppose $a(f+g) + b(f+h) + c(g+h) = 0$. Then $(a+b)f + (a+c)g + (b+c)h = 0$. But by (a), $\{f, g, h\}$ is l.i., so

$$\begin{array}{l} a + b \quad\quad = 0 \\ a \quad\quad + c = 0 \\ \quad\; b + c = 0 \end{array}; \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \approx \cdots \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So $a = b = c = 0$. Hence $\{f+g, f+h, g+h\}$ is l.i.

c) By $(*)$, $\{f, g, h\}$ spans W. By (a), $\{f, g, h\}$ is l.i. $\therefore \{f, g, h\}$ is a basis of W. Thus $\underline{\dim W = 3}$. Since (b) shows that the $\underline{3}$ vectors $f+g, f+h, g+h$ are l.i., they are a basis of W as well.

Hence $\{f, g, h\}$ & $\{f+g, f+h, g+h\}$ are 2 different bases

d) $\dim W = 3$, as is shown in (c)

6. Consider the folllowing subsets of $\mathbf{R}^4$:

$$U = \{(a, a+b, a-b, b) \in \mathbf{R}^4 \mid a, b \in \mathbf{R}\}$$

$$V = \{(a, b, c, d) \in \mathbf{R}^4 \mid a+b+c+d = 0\}$$

a) Show that U is a subspace of $\mathbf{R}^4$, without using the subspace test.

b) Show that V is a subspace of $\mathbf{R}^4$, without using the subspace test.

c) Find a basis of V and hence give its dimension.

a) $U = \text{span}\{(1,1,1,0), (0,1,-1,1)\}$ and hence is a subspace. (Every span is a subspace).

b) V is the set of all solutions of a homogeneous linear system, and is hence a subspace of $\mathbb{R}^4$. (OR: $V = \ker [1\ 1\ 1\ 1]$, and hence is a subspace.)

c) $\begin{array}{cccc|c} & r & s & t & \\ \left[1\right. & 1 & 1 & 1 & \left.0\right] \end{array}$ $\therefore$ $\begin{array}{l} a = -r-s-t \\ b = r \\ c = s \\ d = t \end{array}$; $r, s, t \in \mathbb{R}$

The fundamental solutions are then $v_1 = (-1,1,0,0)$, $v_2 = (-1,0,1,0)$ and $v_3 = (-1,0,0,1)$. Hence, $\{v_1, v_2, v_3\}$ is a basis of W.

Thus, $\dim W = 3$.