1341A.F02 Mini test 1 Solutions

1. Which two of the following statements are true?

I. The span of two distinct vectors u and v in $\mathbf{R}^3$ is a plane through the origin.
False: e.g. span$\{(0,0,0),(1,0,0)\}$ is a line through the origin

II. The span of a single vector u in $\mathbf{R}^2$ is a line.
False: span$\{(0,0)\} = \{(0,0)\}$ is a point

III. A set of vectors $\{u, v, w\} \subseteq X$ spans a vector space X if every $x \in X$ is a linear combination of v and w. True: then $x = 0u + av + bw$ for some $a, b \in \mathbb{R}$

IV. $\{(1,0),(1,1)\}$ spans $\mathbf{R}^2$. True: $(x,y) = (x-y)(1,0) + y(0,1)$

V. $\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\}$ spans $\mathbf{M}_{22}$. False: $\text{span}\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$

$= \left\{\begin{bmatrix}a&b\\c&c\end{bmatrix} \,\middle|\, a,b,c \in \mathbb{R}\right\}$

does not include the matrix $\begin{bmatrix}0&0\\1&0\end{bmatrix}$, for example.

A. I & III
B. II & IV
C. I & II
D. III & IV
E. III & II
F. I & V

2. Which of the following are vector spaces?

(1) $\{(x,y,z) \in \mathbf{R}^3 \mid 2x - 3y + z = 0\}$, together with the usual vector operations of $\mathbf{R}^3$. ✓

(2) $\{(x,y,z) \in \mathbf{R}^3 \mid xyz = 0\}$, together with the usual vector operations of $\mathbf{R}^3$. ×

(3) $\{\begin{bmatrix}0 & a\\b & c\end{bmatrix} \in \mathbf{M}_{22} \mid a,b,c \in \mathbf{R}\}$, together with the usual vector operations of $\mathbf{M}_{22}$ ✓

A. (3) only
B. (2) only
C. (2) and (3)
D. (1) and (2)
E. (1) only
F. (1) and (3)

(1) is a plane through the origin, and so is a subspace of $\mathbb{R}^3$ and $\therefore$ is a v.s.

(2) is not closed under addn: e.g. $(1,0,0)$ and $(0,1,1)$ belong but their sum $(1,0,0)+(0,1,1)=(1,1,1)$ does not.
$\therefore$ (2) is not a v.s.

(3) this is $\text{span}\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$, and so is a subspace of M_{22}, and so is a v.s.

3. Which of $U = \{\begin{bmatrix} x & x \\ y & x-y \end{bmatrix} \in \mathbf{M}_{22} \mid x,\ y \in \mathbf{R}\}$, $V = \{\begin{bmatrix} x & x+y \\ y & y \end{bmatrix} \in \mathbf{M}_{22} \mid x,\ y \in \mathbf{R}\}$ and $W = \{\begin{bmatrix} x & x \\ y & xy \end{bmatrix} \in \mathbf{M}_{22} \mid x,\ y \in \mathbf{R}\}$ are subspaces of $\mathbf{M}_{22}$?

A. U and V only
B. U and W only
C. V and W only
D. U only
E. V only
F. W only

$U = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ and so is a subspace of M_{22}.

$V = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ and so is a subspace of M_{22}

W is not closed under addition : e.g.

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in W$ but

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \notin W$

Hence, W is not a subspace.

4. Which two of the following are **not** subspaces of $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) = 0 \text{ or } f(3) = 0\}$
$T = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-x) = -f(x), \forall x \in \mathbf{R}\}$ (all odd functions)
$U = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) > 0\}$
$V = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) = 0\}$

A. V and T.
B. T and U.
C. S and T.
D. V and S.
E. S and U.
F. V and U.

S is not a subspace : e.g. $f(x) = x - 1 \in S$, $g(x) = x - 3 \in S$ but $(f+g)(x) = 2x - 4 \notin S$.

T is a subspace (see your class notes!)

U is not a subspace, since the zero function $\notin U$

V is a subspace (see your class notes!)

5. Which one of the following spans the subspace $U = \{(x,\ y,\ z) \in \mathbf{R}^3 \mid x - y - z = 0\}$ of $\mathbf{R}^3$?

A. $\{(1,\ 0,\ 0),\ (0,\ 0,\ 1)\}$
B. $\{(1,\ 1,\ 0),\ (0,\ 1,\ 1)\}$
C. $\{(1,\ 0,\ 0)\}$
D. $\{(1,\ 1,\ 0),\ (1,\ 0,\ 1)\}$
E. $\{(1,\ -1,\ -1)\}$
F. $\{1,\ -1,\ -1),\ (1,\ 0,\ 1)\}$

If (x, y, z) satisfies $x - y - z = 0$,

then $(x, y, z) = (y+z, y, z)$, so

$U = \{(y+z, y, z) \mid y, z \in \mathbb{R}\}$

$= \text{span}\{(1,1,0), (1,0,1)\}$

6. Let $v = (0, 1, -2)$ and $U = \{u \in \mathbf{R}^3 \mid \operatorname{proj}_v u = 0\}$

a) Show that if $u = (x, y, z) \in \mathbf{R}^3$, then $\operatorname{proj}_v u = \frac{y-2z}{5}(0, 1, -2)$

b) Find a cartesian equation for U.

c) Give a complete geometric description of U. Is U a subspace of $\mathbf{R}^3$?

d) Find a spanning set for U.

a) $\operatorname{proj}_v (x,y,z) = \left(\frac{(x,y,z)\cdot v}{\|v\|^2}\right)v = \frac{y-2z}{5}(0,1,-2)$

1 1

b) $U = \{(x,y,z) \mid \frac{y-2z}{5}(0,1,-2) = (0,0,0)\} = \{(x,y,z) \mid \underbrace{y-2z=0}_{\frac{1}{2}}\}$

1 (Since $\frac{y-2z}{5}(0,1,-2) = 0 \iff y-2z=0$.) $\frac{1}{2}$

because $(0,1,-2) \neq 0$. ½

c) From (b), U is the plane through O with normal $(0,1,-2)$. ½ ½

1 ½ Yes, because U is a plane through O. ½

!! d) (x,y,z) satisfies $y-2z=0$ iff

$1\frac{1}{2}$ $(x,y,z) = (x, 2z, z) = x(1,0,0) + z(0,2,1)$

$\frac{1}{2}$ - justification

$\therefore$ $U = \operatorname{span}\{(1,0,0), (0,2,1)\}$

$\therefore$ $\{\underbrace{(1,0,0), (0,2,1)}_{1}\}$ spans U.

7. Consider the vector space $\mathbf{F}(\mathbf{R}) = \{f \mid f\colon \mathbf{R} \to \mathbf{R}\}$, with the standard operations. Recall that the zero of $\mathbf{F}(\mathbf{R})$ is the function that has the value 0 for all $x \in \mathbf{R}$.

Let $W = \{f \in \mathbf{F}(\mathbf{R}) \mid f(x) = f(-x) \text{ for all } x \in R\}$ be the subset of even functions.

a) Show that W is a subspace of $\mathbf{F}(\mathbf{R})$.

b) If $f(x) = 1$ is constant, $g(x) = \sin^2 x$ and $h(x) = \sin x$, show that $f, g \in W$ but $h \notin W$.

c) Explain why $U = \operatorname{span}\{1, \sin^2 x\}$ is a subspace of W, without using the subspace test.

a) We apply the subspace test (the abbreviated version):

2

1) $0(x) = 0 = -0 = -0(x)$, $\forall x \in \mathbb{R}$, so $0(x) \in W$ $\frac{1}{2}$

2) If $f, g \in W$ and $k \in \mathbb{R}$, then

1 $\left\{ \begin{array}{l} (f+kg)(x) = f(x) + kg(x) = f(-x) + kg(-x) \quad \text{(since } f,g \in W) \\ \qquad = (f+kg)(-x), \quad \forall x \in \mathbb{R} \\ \therefore f+kg \in W. \end{array} \right.$

$\therefore W$ is a subspace of $F(\mathbb{R})$.

$\frac{1}{2}$ - clarity

b) Clearly, $f(x) = 1 = f(-x)$, $\forall x \in \mathbb{R}$, so $f \in W$ $\frac{1}{2}$

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$g(x) = \sin^2 x = (-\sin x)^2 = [\sin(-x)]^2 = g(-x)$, $\forall x \in \mathbb{R}$ $\frac{1}{2}$

$\therefore g \in W$.

However, $\sin(\frac{\pi}{2}) = 1 \neq -1 = \sin(-\pi/2)$, so $h \notin W$. 1

c) By (b), $1, \sin^2 x \in W$ so $U = \operatorname{span}\{1, \sin^2 x\} \subseteq W$, 1 because

2 W is a subspace. Moreover, since is a span, U is a subspace 1

of W.