1. The eigenvalues of the matrix $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 2 & -3 \end{bmatrix}$ are:

A. 2, 3, 4
B. -3, 3, 4
C. 0, 1, 3
D. -3, 0, 4
E. -1, -2, 1
F. 0, 0, 2

$$\begin{vmatrix} \lambda-1 & -1 & 1 \\ 0 & \lambda & 1 \\ 0 & -2 & \lambda+3 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda & 1 \\ -2 & \lambda+3 \end{vmatrix}.$$

$$= (\lambda-1)(\lambda+1)(\lambda+2)$$

$$\therefore \lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = -2$$

2. Find the main diagonal of the inverse of $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 2 & 4 \\ -3 & 0 & 2 \end{bmatrix}$.

A. $(2,\ -7/2,\ -1)$
B. $(5/2,\ 7/2,\ 3/2)$
C. $(2,\ 1,\ -1)$
D. $(-1,\ -7/2,\ 3)$
E. $(7/2,\ 2,\ -1)$
F. $(2,\ 1,\ -7/2)$

$$[A \mid I_3] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 1 & 0 & -4 & -\frac{7}{2} & 1 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & 2 & -1 \\ -4 & -\frac{7}{2} & 1 \\ 3 & 3 & -1 \end{bmatrix}$$

3. Solve the following equation for A:

$$A^t - [1\ 0\ 0]^t[0\ 1] = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

A. $\begin{bmatrix} 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ B. $\begin{bmatrix} 0 & 2 & 3 \\ 4 & 4 & 6 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 6 \end{bmatrix}$ E. $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ F. $\begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$

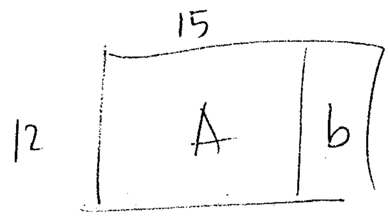
$$A^t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}[0\ \ 1] = \begin{bmatrix} 1 & 4 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 6 \end{bmatrix}$$

4. For a non-homogeneous system of 12 equations in 15 unknowns, answer the following three questions:

- Can the system be inconsistent? Yes
- Can the system have infinitely many solutions? Yes
- Can the system have a unique solution? No

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.



$\text{rank} A \leq 12$

$\#\text{parameters} = 15 - \text{rank} A \geq 3$

5. Let $U = \text{span}\{(1,-2,3,4), (-3,6,-5,-16), (-1,2,-5,-2)\}$ be a subspace of $\mathbf{R}^4$. Then $\dim U^{\perp}$ is

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

$$U^{\perp} = \ker \underbrace{\begin{bmatrix} 1 & -2 & 3 & 4 \\ -3 & 6 & -5 & -16 \\ -1 & 2 & -5 & -2 \end{bmatrix}}_{A} \longrightarrow \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dim U = \dim \ker A = 4 - \operatorname{rank} A = 4 - 2 = 2$$

6. The dimension of $S = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22} \mid A = A^t\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A = A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Longrightarrow b = c$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$ is a basis of S.

(check it)
(is l.i.)

$\therefore \dim S = 3$.

7. Which of the following statements are true?

(1) Each spanning set for $\mathbb{R}^n$ has exactly n vectors. false (can have $> n$)
(2) If $\{u, v, w\}$ is linearly independent, then $\{u, v\}$ is also linearly independent. true
(3) If A is an $n \times n$ matrix, then $\det A = (-1)^n \det(A^t)$. false ($\det A = \det A^t$)
(4) If A is an $n \times n$ matrix, then $\dim \operatorname{col} A = n$. false, $\dim \operatorname{col} A = \operatorname{rank} A \leq n$
(5) If A is an $n \times n$ matrix, the $\dim \ker A = n - \operatorname{rank} A$. true
(6) The set of $n \times n$ diagonal matrices is a subspace of the vector space of of all $n \times n$ matrices. true

A. All six are true.
B. (2), (5) and (6).
C. (1), (2) and (4).
D. (3), (2) and (6).
E. (4), (5) and (6).
F. (2), (3) and (5).

8. Let A and B be two matrices such that $\det A = -2$ and $\det(B^t) = 3$. Find $\det(AB)$.

A. 6
B. 1
C. 5
D. -5
E. -6
F. -3/2

$$\det(AB) = (\det A)(\det B)$$
$$= (\det A)(\det B^t)$$
$$= (-2)3 = -6$$

9. What is the dimension of the subspace U of $\mathbf{R}^3$ spanned by $(1,\ 1,\ 1)$, $(-1,\ 1,\ -1)$, $(1,\ 1,\ 3)$ and $(0,\ 2,\ 1)$?

A. 0
B. 1
~~C. 2~~
D. 3
E. 4
F. These vectors do not span a subspace.

$$U = \text{row} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}}_{A} = \text{row} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \text{row} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \dim U = \operatorname{rank} A = 3.$$

10. Consider the linear system

$$\begin{array}{rcrcrcl} x & - & y & - & z & = & 1 \\ x & & & + & z & = & p \\ & & y & + & qz & = & 3 \end{array}$$

a) If $[A\,|\,b]$ is the augmented matrix of the system above, find rank A and rank$[A\,|\,b]$ for all values of p and q.

$$[A|b] = \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & p \\ 0 & 1 & q & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 2 & p-1 \\ 0 & 0 & q-2 & 4-p \end{array}\right] \quad \frac{1}{2}$$

$$\text{rank} A = \begin{cases} 2 & \text{if } q = 2 \\ 3 & \text{if } q \neq 2 \end{cases} \quad \frac{1}{2}$$

$$\text{rank}[A|b] = \begin{cases} 2 & \text{if } q = 2 \ \& \ p = 4 \quad \frac{1}{2} \\ 3 & \text{if } q \neq 2 \text{ or } p \neq 4 \quad \frac{1}{2} \end{cases}$$

10b) Find all p and q so that this system has
i) a unique solution,
ii) infinitely many solutions, and
iii) no solutions.

i) If $q \neq 2$, then $\operatorname{rank} A = \operatorname{rank}[A|b] = 3$, 1/2
so the system has a unique solution.

1/2
ii) If $q=2$ & $p=4$, then $\operatorname{rank} A = \operatorname{rank}[A|b] = 2 < 3$
so the system has infinitely many solutions.

1/2
iii) If $q=2$ & $p \neq 4$, then $\operatorname{rank} A = 2 \neq 3 = \operatorname{rank}[A|b]$
so the system has no solution. 1 - reasons

10c) In case (ii) above, give a complete geometric description of the set of solutions.

If $q=2$ & $p=4$, then

$$[A|b] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{aligned} x &= 4 - t \\ y &= 3 - 2t \\ z &= t \end{aligned}$$

1/2

The general solution:
$\Rightarrow (x, y, z) = (4, 3, 0) + t(-1, -2, 1)$, $t \in \mathbb{R}$.

1/2

It is a line through $(4,3,0)$ with the direction $(-1,-2,1)$ 1/2

11. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

a) Find $\det(A - 2I_3)$ and hence conclude that 2 is an eigenvalue of A.

$$|A - 2I_3| = \begin{vmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{vmatrix} = 0$$ since 2 rows are the same.

b) Find a basis of $E_2 = \{x \in \mathbf{R}^3 \mid Ax = 2x\}$.

$$E_2 = \ker(A - 2I) = \ker \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \ker \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$s \quad t$

$x = -s - t$

$y = s \qquad ; s, t \in \mathbb{R}$

$z = t$

$\therefore \{(-1, 1, 0), (-1, 0, 1)\}$ is a bases of E_2

11 c) Check that $(1,1,1)$ is an eigenvector of $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$.

$$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix} = -\begin{bmatrix}1\\1\\1\end{bmatrix},$$ and $(1,1,1) \neq (0,0,0)$.

$\therefore$ $(1,1,1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

1

11 d) If possible, find an invertible matrix P such that $P^{-1}AP = D$ is diagonal and give this diagonal matrix D.

By b) and c), we have $\dim E_2 + \dim E_{-1} = 3$. — $*$

1

Let $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Then P is invertible because of $*$,

1/2

and $$\underbrace{P^{-1}AP = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{1}.$$

12. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x - y - 2z = 0\}$.

a) Find a basis of W and give the dimension of W.

$(x, y, z) = (y+2z, y, z) = y(1,1,0) + z(2,0,1)$

$\therefore W = \text{span}\{(1,1,0), (2,0,1)\}$.

1

$\{(1,1,0), (2,0,1)\}$ is l.i. $((1,1,0) \nparallel (2,0,1))$

$\therefore \{\underset{v_1}{(1,1,0)}, \underset{v_2}{(2,0,1)}\}$ is a basis of W.

1 $\dim W = 2$. (OR: $W = \ker [1\ {-1}\ {-2}]$; $\begin{matrix} x = s + 2t \\ y = s \\ z = t \end{matrix}$, $s, t \in \mathbb{R}$ gives $\{(1,1,0), (2,0,1)\}$

b) Find an orthogonal basis of W.

$w_1 = v_1$; $w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1$ 1

$w_1 = (1,1,0)$,

$w_2 = (2,0,1) - \frac{(2,0,1)\cdot(1,1,0)}{1^2 + 1^2}(1,1,0)$

$= (2,0,1) - (1,1,0) = (1,-1,1)$.

$\therefore \{w_1, w_2\}$ is an orthogonal basis of W.

1

12c) Find the best approximation to $(1, 0, 0)$ by vectors in W.

The best approximation to $(1,0,0)$ by vectors in W is $\text{proj}_W(1,0,0)$, where

1

$$\text{proj}_W(1,0,0) = \frac{(1,0,0)\cdot(1,1,0)}{1+1}(1,1,0) + \frac{(1,0,0)\cdot(1,-1,1)}{1+1+1}(1,-1,1)$$

$\frac{1}{2}$

$$= \left(\frac{1}{2}, \frac{1}{2}, 0\right) + \left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(\frac{5}{6}, \frac{1}{6}, \frac{1}{3}\right).$$

$\frac{1}{2}$

13. a) Let A be a real $n \times n$ matrix. Give 4 additional different statements which are equivalent to:

$$Ax = b \text{ is consistent for all } b \in \mathbf{R}^n.$$

(I)

$\det A \neq 0$

1 A^t is invertible

$A \sim I_n$

(II) rank $A = n$

1

(III) $Ax = 0 \Leftrightarrow x = 0$

etc.

1

(IV) the columns of A are a basis of $\mathbb{R}^n$.

" are l.i

1 " span $\mathbb{R}^n$

the rows of A span $\mathbb{R}^n$

" are l.i.

" are a basis of $\mathbb{R}^n$

13b) State (in the box) whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

i) If a 5 by 3 matrix A has rank 3, the system $Ax = 0$ has a unique solution. $\boxed{\text{true}}$

$\operatorname{rank} A = 3$, so

$\dim(\ker A) = 3 - \operatorname{rank} A = 3 - 3 = 0$.

$\therefore$ $Ax = 0$ has a unique solution $x = 0$.

ii) If a 5 by 5 matrix A satisfies $A^3 = 0$, but $A \neq 0$, then A is invertible. $\boxed{\text{false}}$

$A^3 = 0 \Rightarrow (\det A)^3 = 0 \Rightarrow \det A = 0$, so A is <u>not</u> invertible

Ex. $A = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$,

$A^2 = \begin{bmatrix} B^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $A^3 = 0$, $A \neq 0$,

but A is not invertible (directly: $\operatorname{rank} A = 1 < 5$).

14. Let $u = (1,0,0,1), v = (0,1,1,0)$ and $w = (1,0,0,-1)$ be vectors in $\mathbf{R}^4$.

a) Carefully show, by any method, that $\{u, v, w\}$ is linearly independent.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

rank $A = 3$, $\therefore \{u, v, w\}$ is l.i. 2

OR: $u \neq 0$, $v \neq 0$, $w \neq 0$, $u \cdot v = 0 = u \cdot w = v \cdot w$, so $\{u, v, w\}$ is orthogonal and hence l.i.

OR: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which has rank 3. $\therefore \{u, v, w\}$ is l.i.

b) Find a vector $z \in \mathbf{R}^4$ so that $\{u, v, w, z\}$ spans $\mathbf{R}^4$, and support your answer.

Let $\underline{Z = (0, 0, 1, 0) \in R^4}$. 1

$$B = \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

1 – reason

rank $B = 4$, $\therefore \{u, v, w, z\}$ is l.i.

$\therefore \{u, v, w, z\}$ is a basis of R^4 because $\dim R^4 = 4$. So $\{u, v, w, z\}$ spans R^4.

14c) If $(x_1, x_2, x_3, x_4) \in \text{span}\{u, v, w\}$, find formulae in terms of x_1, x_2, x_3 and x_4 for the scalars c_1, c_2 and c_3 such that $\underbrace{(x_1, x_2, x_3, x_4)}_{X} = c_1 u + c_2 v + c_3 w$.

Since $\{u, v, w\}$ is orthogonal, by the Expansion thm, $c_1 = \frac{X \cdot u}{\|u\|^2}$, $c_2 = \frac{X \cdot v}{\|v\|^2}$, $c_3 = \frac{X \cdot w}{\|w\|^2}$,

1

So $c_1 = \frac{x_1 + x_4}{2}$

$c_2 = \frac{x_2 + x_3}{2}$

& $c_3 = \frac{x_1 - x_4}{2}$

1 (2 of 3)

OR: see next page

14c) If $(x_1, x_2, x_3, x_4) \in \text{span}\{u, v, w\}$, find formulae in terms of x_1, x_2, x_3 and x_4 for the scalars c_1, c_2 and c_3 such that $(x_1, x_2, x_3, x_4) = c_1u + c_2v + c_3w$.

Let $[A \mid b] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 1 & 0 & x_3 \\ 1 & 0 & -1 & x_4 \end{array}\right]$.

Then $[A \mid b] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{x_1+x_4}{2} \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 0 & x_3 - x_2 \\ 0 & 0 & 1 & \frac{x_1-x_4}{2} \end{array}\right]$

If $(x_1, x_2, x_3, x_4) \in \text{span}\{u, v, w\}$, then the system $[A \mid b]$ has solutions, so $x_3 - x_2 = 0$. The unique solution is (c_1, c_2, c_3) with

$$c_1 = \frac{x_1 + x_4}{2}, \quad c_2 = x_2, \quad c_3 = \frac{x_1 - x_4}{2}.$$

$((x_1, x_2, x_3, x_4) = c_1u + c_2v + c_3w$ if $x_2 = x_3.)$