1. Consider the following subsets of $\mathbf{R}^2$: 134B Mid Term W02

$U = \{(x,y) \in \mathbf{R}^2 \mid xy = 0\}$, $V = \{(x,y) \in \mathbf{R}^2 \mid x \geq 0, y \geq 0\}$, and $W = \{(x,y) \in \mathbf{R}^2 \mid x+y = 0\}$.

Which of the following statements are true?

I. U is closed under multiplication by scalars.
II. U is closed under addition.
III. V is closed under multiplication by scalars.
IV. V is closed under addition.
V. $\dim W = 2$

A. III & II
B. I & III
C. II & IV
D. III & V
E. I & IV
F. I & V

I is true: if $a \in R$, $(x,y) \in U$, then $xy = 0$, so

$$(ax)(ay) = a^2(xy) = 0. \text{ Thus } a(x,y) = (ax, ay) \in U.$$

II is false: $(1,0) \in U$, $(0,1) \in U$, but

$$(1,0)+(0,1) = (1,1) \notin U.$$

III is false: $(1,1) \in V$, but $-(1,1) = (-1,-1) \notin V$.

IV is true: If $(x_1, y_1), (x_2, y_2) \in V$, then $x_1 \geq 0$, $x_2 \geq 0$, $y_1 \geq 0$, $y_2 \geq 0$. So $x_1 + x_2 \geq 0$, $y_1 + y_2 \geq 0$. Thus $(x_1, y_1) + (x_2, y_2)$ $= (x_1 + x_2, y_1 + y_2) \in V$.

V is false: $W = \operatorname{span}\{(1,-1)\}$, so $\dim W = 1$.

2. Suppose X is a subspace of $\mathbf{R}^6$, that $X \neq \{0\}$ and that $X \neq \mathbf{R}^6$. Which of the following statements are true?

I. X has a spanning set consisting of 6 vectors.
II. X has a linearly independent subset consisting of 6 vectors.
III. $1 \leq \dim X \leq 5$.
IV. X has a basis that spans $\mathbf{R}^6$.
V. For all vectors u, v, w in X, $au + bv + cw = 0$ implies $a = b = c = 0$.

A. III & II
B. I & III
C. II & IV
D. III & V
E. I & IV
F. I & V

I is true: Since $\dim X \leq 6$, every basis of X can be enlarged to a spanning set of X consisting of 6 vectors.

II is false: Since $X \neq R^6$, $\dim X < 6$. So every subset of X which consists of 6 or more vectors is l.d.

III is true: Since $\dim X < 6$ and $X \neq \{0\}$, $1 \leq \dim X \leq 5$.

IV is false: because $\dim X \leq 5$ and every spanning set of R^6 contains at least 6 vectors.

V is false: e.g. choose $w = u + v$, then

$$1u + 1v + (-1)w = u + v - (u+v) = 0.$$

3. Let $W = \{(a, a-b, a+b) \mid a, b \in \mathbf{R}\}$.

1 a) By any method, show that W is a subspace of $\mathbf{R}^3$.

$2\frac{1}{2}$ b) Find a basis of W and give the dimension of W.

1 c) Give a geometric description of W.

1 d) Find an equation for W.

a) $(a, a-b, a+b) = a(1,1,1) + b(0,-1,1)$, so

1 $W = \text{span}\{(1,1,1), (0,-1,1)\}$ is a subspace of R^3.

(or by the subspace test to prove it.)

b) $\{(1,1,1), (0,-1,1)\}$ is a basis of W, because

it spans W and it is l.i. $((1,1,1) \not\parallel (0,-1,1))$.

So $\dim W = 2$ ½ 1 - mentioning "spans", "li"

½ - showing spans ½ l.i.

c) W is a plane ½ through the origin ½ with a normal

$n = (1,1,1) \times (0,-1,1) = (2,-1,-1)$

½ (this doesn't need to be correct)

d) An equation for W (as the plane):

$(2,-1,-1) \cdot (x,y,z) = 0$, that is,

$2x - y - z = 0$.

½ correct ½ consistent with (c)

4. Let $v_1 = (-1,1,1,1), v_2 = (1,-1,1,1), v_3 = (1,1,-1,1)$, and let W be the subspace of $\mathbf{R}^4$ defined by

$$W = \operatorname{span}\{v_1, v_2, v_3\}.$$

1 a) Show that $\{v_1, v_2, v_3\}$ is an orthogonal set.

1 b) Find a basis of W and hence find $\dim W$.

1 c) The vector $u = (-4,6,0,2)$ belongs to W. Find $c_1, c_2, c_3 \in \mathbf{R}$ such that $u = c_1v_1+c_2v_2+c_3v_3$.

$1\frac{1}{2}$ d) Show that $\{v_1 + 2v_2, v_1 + 3v_3, v_2 + v_3\}$ is a basis of W. you do not need to compute these [illegible]

$1\frac{1}{2}$ e) Assuming that $v_4 = (1,0,0,0)$ does not belong to W (you do not have to check this), explain why $\{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbf{R}^4$.

a) $v_1 \cdot v_2 = -1-1+1+1=0$, $v_1 \cdot v_3 = 0$, $v_2 \cdot v_3 = 0$ $\frac{1}{2}$

and $v_1 \neq 0$, $v_2 \neq 0$, $v_3 \neq 0$ $\frac{1}{2}$ So $\{v_1, v_2, v_3\}$ is an orthogonal set.

b) Since $\{v_1, v_2, v_3\}$ is orthogonal, it is l.i.. $\frac{1}{2}$

So it is a basis of W. $\dim W = 3$ $\frac{1}{2}$

c) $c_1 = \dfrac{u \cdot v_1}{\|v_1\|^2} = \dfrac{12}{4} = 3$, $c_2 = \dfrac{u \cdot v_2}{\|v_2\|^2} = \dfrac{-8}{4} = -2$,

$\frac{1}{2}$ Correct formula

$c_3 = \dfrac{u \cdot v_3}{\|v_3\|^2} = \dfrac{4}{4} = 1$.

$\frac{1}{2}$ if ≥ 2 are consistent with the form

Correct formula

d) Since $\dim W = 3$, we only need to prove that $\{v_1 + 2v_2, v_1 + 3v_3, v_2 + v_3\}$ is l.i..

$$a(v_1+2v_2)+b(v_1+3v_3)+c(v_2+v_3)=0$$

$$\Rightarrow (a+b)v_1+(2a+c)v_2+(3b+c)v_3=0.$$

1/2

$$\Rightarrow \begin{cases} a+b=0 \\ 2a+c=0 \\ 3b+c=0 \end{cases}$$ (because $\{v_1, v_2, v_3\}$ is l.i.)

1/2

$$\Rightarrow a=b=c=0$$

Thus $\{v_1+2v_2, v_1+3v_3, v_2+v_3\}$ (a subset of W) is l.i.,

and hence it is a basis of W.

e) Since $\{v_1, v_2, v_3\}$ is l.i., and

$v_4 \notin \text{span}\{v_1, v_2, v_3\}$, by the Independent Lemma

that $\{v_1, v_2, v_3, v_4\}$ is l.i. 1/2 Saying this 1/2 justification of it

Since $\dim R^4 = 4$, $\{v_1, v_2, v_3, v_4\}$ is a basis

of R^4. 1/2

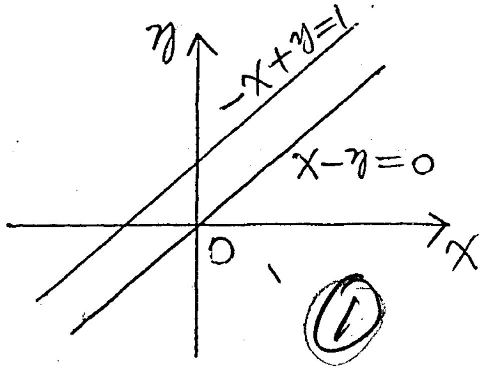
5. In each case, give an explicit example of:

a) An inconsistent linear system of 2 equations in 2 unknowns. Sketch the graphs of the equations of your system to illustrate your example.

b) A linear system of 2 equations in 3 unknowns with at least 2 different solutions. Give two different solutions.

c) A linear system of 4 equations in 4 unknowns with a unique solution. Give the unique solution.

a) $\begin{cases} x - y = 0 \\ -x + y = 1 \end{cases}$ is inconsistent. 1



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b) $\begin{cases} x + y + z = 1 \\ \quad\;\; y + z = 0 \end{cases}$ has at least two different 1

solutions: $(x, y, z) = (1, 0, 0)$ and
$(x, y, z) = (1, 1, -1)$. 1

c) $\begin{cases} x_1 + x_2 = 3 \\ x_2 + x_3 = 2 \\ x_3 + x_4 = 0 \\ x_4 = 1 \end{cases}$ has a 1

unique solution: $(x_1, x_2, x_3, x_4) = (0, 3, -1, 1)$.

1