1. Which one of the following spans the subspace $\{(x,\ y,\ z) \in \mathbf{R}^3 \mid 2x - y + 3z = 0\}$ of $\mathbf{R}^3$?

A. $\{(3,\ 0,\ -2)\}$
B. $\{(1,\ 0,\ 0),\ (0,\ 1,\ 0)\}$
C. $\{(1,\ 2,\ 0)\}$
D. $\{(-3,\ 0,\ 2),\ (1,\ 0,\ 0)\}$
E. $\{(1,\ 2,\ 0),\ (0,\ 3,\ 1)\}$
F. $\{(1,\ 0,\ 0),\ (1,\ 2,\ 0)\}$

Let $U=\{(x,y,z)\in R^3 \mid 2x-y+3z=0\}$.
U is a plane through the origin.
The set in A or C only spans a line.
Because $(1,0,0)\notin U$, every set in B, D or F cannot span U.

$(1,2,0)\times(0,3,1)=(2,-1,3)$, so
$\text{span}\{(1,2,0),(0,3,1)\}=U$.

OR: $U=\{(x,2x+3z,z) \mid x,z\in \mathbb{R}\} = \text{Span}\{(1,2,0),(0,3,1)\}$.

2. Which two of the following are not subspaces of $\mathbf{R}^4$?
$V = \{(a,\ b,\ c,\ d) \mid c = a + 2b,\ d = a - 3b\}$
$S = \{(a,\ b,\ c,\ d) \mid a = 0,\ b = 0\}$
$T = \{(a,\ b,\ c,\ d) \mid a - b = 2,\ c = d\}$
$U = \{(a,\ b,\ c,\ d) \mid a \geq 0,\ b \geq 0\}$

A. V and T.
B. T and U.
C. S and T.
D. V and S.
E. S and U.
F. V and U.

V: $(a,b,c,d)=a(1,0,1,1)+b(0,1,2,-3)$
$\therefore V=\text{span}\{(1,0,1,1),(0,1,2,-3)\}$, a s.s
$S=\text{span}\{(0,0,1,0),(0,0,0,1)\}$, a subspace.
T: $(0,0,0,0)\notin T$, not a subspace
U: $(1,0,0,0)\in T$, but $-(1,0,0,0)\notin T$
not a subspace.

3. Which of the following statements are true?

I. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbb{R}$, $au + bv + cw = 0$ implies $a = b = c = 0$.

II. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbb{R}$, $au + bv + cw = 0$ when $a = b = c = 0$.

III. A set $\{u, v, w\}$ of vectors is linearly independent iff u is not a linear combination of v and w.

IV. $\{(1,0), (1,1)\}$ spans $\mathbb{R}^2$.

V. $\{(1,0,1), (1,1,1), (2,1,2)\}$ spans $\mathbb{R}^3$.

A. III & II
B. I & II
C. II & IV
D. III & V
E. I & IV
F. I & V

I : true

II : false, because for any vectors u, v, w, we always have $au+bv+cw=0$ when $a=b=c=0$.

III : false. e.g. $\{(1,0,0),(0,1,0),(0,0,0)\}$ is l.d. but $(1,0,0)$ is not a l.c. of $(0,1,0)$ and $(0,0,0)$.

IV : true. $\forall (x,y) \in R^2$, $(x,y) = (x-y)(1,0) + y(1,1)$.
(OR: $(1,0)$, $(1,1)$ aren't parallel...)

V : false, because $(1,0,0) \notin \operatorname{span}\{(1,0,1),(1,1,1),(2,1,2$

4. For what value of α is the set of vectors $\{(1, 1, 1), (1, 0, 2), (1, \alpha, 1)\}$ linearly dependent?

A. -2
B. -1/2
C. -1
D. 0
E. 1
F. 2

$$a(1,1,1)+b(1,0,2)+c(1,\alpha,1)=(0,0,0)$$

$$\Longleftrightarrow \begin{cases} a+b+c=0 \\ a \quad +\alpha c=0 \\ a+2b+c=0 \end{cases} \Longleftrightarrow \begin{cases} b=0 \\ a=-c \\ (\alpha-1)c=0 \end{cases} \overset{\text{if } \alpha\neq 1}{\Longleftrightarrow} \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

Thus, if $\alpha \neq 1$, then $\{(1,1,1),(1,0,2),(1,\alpha,1)\}$ is l.i.

When $\alpha = 1$, it is clear that the set is l.d.

5. Which of $U = \{(x,\ y,\ x-y) \mid x,\ y \in \mathbf{R}\}$, $V = \{(x,\ y,\ x+y) \mid x,\ y \in \mathbf{R}\}$ and $W = \{(x,\ y,\ xy) \mid x,\ y \in \mathbf{R}\}$ are subspaces of $\mathbf{R}^3$?

A. U only
B. V only
C. W only
D. U and V only
E. U and W only
F. V and W only

U: $(x, y, x-y) = x(1,0,1) + y(0,1,-1)$

$U = \text{span}\{(1,0,1), (0,1,-1)\}$ is a subspace.

$V = \text{span}\{(1,0,1), (0,1,1)\}$ is a subspace.

W: $(1,1,1) \in W$, but $2(1,1,1) = (2,2,2) \notin W$

So W is not a subspace.

6. Let $u = (1, 0, -1)$, and $W = \{w \in \mathbf{R}^3 \mid \text{proj}_u w = 0\}$

$1\frac{1}{2}$ a) Is W a subspace of $\mathbf{R}^3$?

1 b) Show that $\{(1,0,1), (0,1,0\}$ spans W.

$1\frac{1}{2}$ c) Give a geometric description of W.

2 d) If $L = \{v \in \mathbf{R}^3 \mid v \cdot w = 0,\ \text{for all } w \in W\}$, give a a geometric description of L. Is L a subspace of $\mathbf{R}^3$?

a) $W = \{ w \in \mathbb{R}^3 / \left(\frac{w \cdot u}{\|u\|^2}\right) u = 0 \} = \{ w \in \mathbb{R}^3 / w \cdot u = 0 \}$ (Since $u \neq 0$). This is a plane through the origin, and so is a subspace of $\mathbb{R}^3$. (OR: use the subspace test) $(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})$ subspace test thing

b) $W = \{ (x, y, z) \mid x - z = 0 \} = \{ (x, y, x) / x, y \in \mathbb{R} \}$

$= \text{span}\{ (1,0,1), (0,1,0) \}$. 1

c) W is the plane through O with normal $u = (1,0,-1)$. $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

d) $L = \{ v \in \mathbb{R}^3 \mid v \cdot w = 0 \text{ for all } w \in W \}$

$= \{ v \in \mathbb{R}^3 \mid v \cdot (1,0,1) = 0 \text{ and } v \cdot (0,1,0) = 0 \}$

$\frac{1}{2}$ $= \{ (x,y,z) \mid x + z = 0 \text{ and } y = 0 \}$

– some justification for

$= \{ (x, 0, -x) \mid x \in \mathbb{R} \} = \text{span}\{ (1, 0, -1) \}$.

L is the line through O with direction $(1,0,-1)$. 1

Thus, L is a subspace of $\mathbb{R}^3$. $\frac{1}{2}$

7. Let $U = \text{span}\{(1,0,0,1)\}$ and $V = \text{span}\{(0,1,1,-1)\}$ be subspaces of $\mathbf{R}^4$, and let

$$W = \{u + v \in \mathbf{R}^4 \mid u \in U \text{ and } v \in V\}$$

3 a) Show that W is a subspace of $\mathbf{R}^4$.

1 b) Is $(2,3,3,-1)$ in W ?

1 c) Is the intersection $U \cap V = \{\alpha \in \mathbf{R}^4 \mid \alpha \in U \text{ and } \alpha \in V\}$ a subspace of $\mathbf{R}^4$?

1 d) Is the union $U \cup V = \{\alpha \in \mathbf{R}^4 \mid \alpha \in U \text{ or } \alpha \in V\}$ a subspace of $\mathbf{R}^4$?

a) Denote $u_0 = (1,0,0,1)$ and $v_0 = (0,1,1,-1)$. Then,

$W = \{ s u_0 + t v_0 \mid s, t \in \mathbb{R}\}$

2 $= \text{span}\{u_0, v_0\}$, and so is a subspace of $\mathbb{R}^4$.

(OR: use the subspace test.)
OR: (1+1+1)

b) Yes, since $(2,3,3,-1) = 2u_0 + 3v_0$.
$\frac{1}{2}$ $\frac{1}{2}$

c) $0 \in U \cap V$ because $0 \in U$ and $0 \in V$. If $w_1, w_2 \in U \cap V$, then $w_1 + w_2 \in U$ (because $w_1, w_2 \in U$, and U is a subspace) and $w_1 + w_2 \in V$ (for similar reasons). Hence, $w_1 + w_2 \in U \cap V$. If $w \in U \cap V$ and $k \in \mathbb{R}$, then $kw \in U$ ($\because w \in U$ and U is a subspace), and $kw \in V$ (similar reason), so $kw \in U \cap V$. By the subspace test, $U \cap V$ is a subspace of $\mathbb{R}^4$. (OR: $w \in U \cap V \Leftrightarrow w = s u_0 = t v_0$ for some s, t. But u_0 is not a multiple of v_0, and v_0 is not a multiple of u_0, so $s = t = 0$ i.e. $w = 0$. So $U \cap V = \{(0,0,0,0)\}$, which is a subspace of $\mathbb{R}^4$.)

$\frac{1}{2}$ - yes
$\frac{1}{2}$ - jus[tification]

d) No, since $u_0 \in U \cup V$ and $v_0 \in U \cup V$ but $u_0 + v_0 \notin U$ and $u_0 + v_0 \notin V$ so $u_0 + v_0 \notin U \cup V$; thus $U \cup V$ is not closed under addition.

$\frac{1}{2}$ - "[no]"
$\frac{1}{2}$ - jus[tification]