M12 W02 Solutions

1. Which of the following are bases for $\mathbf{R}^3$?

(1) $\{(-1,\ 2,\ 3),\ (3,\ 3,\ 2)\}$ impossible

(2) $\{(1,\ 6,\ 5),\ (1,\ 4,\ 1),\ (1,\ 3,\ -1)\}$

(3) $\{(-1,\ 3,\ -5),\ (1,\ -2,\ 4),\ (2,\ 0,\ 4),\ (5,\ 1,\ 9)\}$ impossible

A. All three are.
B. (1) only.
C. (2) only.
D. (1) and (2).
E. (2) and (3).
F. None of them is.

$\dim R^3 = 3$, so there are 3 vectors in a basis for R^3.

(2) is a basis for $R^3 \iff$ it is l.i. $\iff \operatorname{rank} A = 3$,

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 4 & 3 \\ 5 & 1 & -1 \end{bmatrix}$.

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & -4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \operatorname{rank} A = 2$, so (2) is not a basis for R^3.

2. Examine the following matrices and select the correct statement below.

1) $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ 4) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 5) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

A. 5 and 3 are in row-echelon form
B. 5 and 1 are in row-echelon form
C. 2 and 3 are in reduced row-echelon form
D. Only 3 is in reduced row-echelon form
E. 2 and 4 are in reduced row-echelon form
F. All the matrices are in are in row-echelon form

5 is not in row-echelon form

4 is not in reduced row-echelon form.

3. Find the value(s) of t for which (2, 6, 5, $2t$) lies in the subspace of $\mathbf{R}^4$ spanned by (1, 2, 2, 2), (3, 7, 6, 6) and (1, 2, 1, 2).

A. t = -4 only
B. t = -2 or -4
C. t = 0 or 2
D. t = -2, 0 or 4
E. t = 2 or 4
F. t = 2 only

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 7 & 2 & 6 \\ 2 & 6 & 1 & 5 \\ 2 & 6 & 2 & 2t \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2t-4 \end{array}\right]$$

$AX=b$ is consistent $\Longleftrightarrow$ rank A = rank $[A|b]$
$\Longleftrightarrow 2t-4=0 \Longleftrightarrow t=2$.

4. A basis for the solution space of the system

$$\begin{array}{rcrcrcrcl} u & - & 2x & + & 3y & + & 4z & = & 0 \\ u & - & 2x & + & 4y & + & 6z & = & 0 \end{array}$$

is:

A. {(2, 1, 0, 0), (1, −3, −4, 1)}
B. {(2, 1, 0, 0), (2, 0, −2, 1)}
C. {(2, 0, −2, 1)}
D. {(0, 0, 0, 0)}
E. {(2, 1, 0, 0)}
F. {(1, 2, 0, 0)}

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 1 & -2 & 4 & 6 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$\begin{aligned} u &= 2s + 2t \\ x &= s \\ y &= -2t \\ z &= t \end{aligned}, \quad s, t \in R.$$

$(u, x, y, z) = s(2, 1, 0, 0) + t(2, 0, -2, 1)$.

(It is easy to check that $(1, -3, -4, 1)$ is not a solution to the system.)

5. If A is an $n \times 2$ matrix and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then the second column of the matrix AB is

A. the same as the first column of B.

B. the sum of the first and the second column of A.

C. not defined unless $n = 2$.

D. the same as the second column of A.

E. the same as the second column of B.

F. the same as the first column of A.

Let A_1 and A_2 be the first and the second column of A, respectively. Then $A = [A_1 \ A_2]$.

$$AB = [A_1 \ A_2] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [A_1 + A_2 \quad A_1]$$

(block multiplication)

So the second column of AB is A_1, the same as the first column of A.

6. Suppose $k \in \mathbf{R}$ and consider the linear system

$$\begin{array}{ccccccc} kx & + & y & + & z & = & 1 \\ x & + & ky & + & z & = & 1 \\ x & + & y & + & kz & = & 1 \end{array}$$

a) If $[\,A\,|\,b\,]$ is the augmented matrix of the system above, find rank A and rank$[\,A\,|\,b\,]$ for all values of k.

b) Find all k so that this system has
  i) a unique solution,
  ii) infinitely many solutions, and
  iii) no solutions.

c) In case (ii) above, give a geometric description of the set of solutions.

$$[A|b] = \left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & K & 1 & 1 \\ 0 & 1-K & K-1 & 0 \\ 0 & 1-K^2 & 1-K & 1-K \end{array}\right] *$$

$$\xrightarrow{\text{if } K \neq 1} \left[\begin{array}{ccc|c} 1 & K & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1+K & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & K & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & K+2 & 1 \end{array}\right] **$$

$$\therefore \text{rank} A = \begin{cases} 1 & \text{if } K=1 \\ 2 & \text{if } K=-2 \\ 3 & \text{if } K \neq 1 \ \& \ K \neq -2 \end{cases} \qquad \begin{matrix} 1/2 \\ 1/2 \\ 1/2 \end{matrix}$$

$$\text{rank}[A|b] = \begin{cases} 1 & \text{if } K=1 \\ 3 & \text{if } K \neq 1 \end{cases} \qquad \begin{matrix} 1/2 \\ 1/2 \end{matrix}$$

(If these are incorrect but all are consistent with * and **, give $\frac{1\frac{1}{2}}{2\frac{1}{2}}$; reduce otherwise. If * and * are correct, but all of the rest is incorrect, give $1/2/2\frac{1}{2}$.

b) i) unique solution $\Longleftrightarrow$ $\operatorname{rank} A = \operatorname{rank}[A|b] = 3$

$\Longleftrightarrow K \neq 1$ & $K \neq -2$. $\frac{1}{2}$

(ii) infinitely many solutions $\Longleftrightarrow$ $\operatorname{rank} A = \operatorname{rank}[A|b] < 3$

$\Longleftrightarrow K = 1$. $\frac{1}{2}$

(iii) no solution $\Longleftrightarrow$ $\operatorname{rank} A \neq \operatorname{rank}[A|b]$

$\Longleftrightarrow K = -2$, $(K \neq 1)$. $\frac{1}{2}$

$+\frac{1}{2}$ - reasons in ≥ 2 cases

c) In case (ii) above, $K=1$, so

$$[A|b] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General solution:

$$(x, y, z) = (1-s-t, s, t)$$
$$= (1,0,0) + s(-1,1,0) + t(-1,0,1),$$
$$s, t \in \mathbb{R}.$$

The set of solutions is

$\{(x, y, z) \mid x+y+z = 1\}$. $\frac{1}{2}$ It is a plane through $(1,0,0)$ $\frac{1}{2}$ with normal vector $(1,1,1)$ $\frac{1}{2}$

(with justification; if incorrect but consistent with (ii) $-1\frac{1}{2}$)

($\frac{1}{2}$) - correct with some but not correct justification, or an example that is incorrect; (1) - both parts correct 7

7. In each case, either show that the statement is true or give an explicit example (with numbers!)to show that it is false. Let A and B denote matrices, and x and b denote column vectors (i.e. $k \times 1$ matrices for some k).

(0) - no justification at all.

a) If A is $m \times n$ and rank $A = m$, then the system $Ax = 0$ has only the trivial solution.

false. e.g. $A = \begin{bmatrix} 1 & -1 \end{bmatrix}_{1\times 2}$, rank $A = 1$

but $AX = 0$ has infinitely many solutions $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t \in R$.

b) If A is $m \times n$ and $m < n$, the system $Ax = b$ is consistent for any $b \in \mathbf{R}^m$

false. e.g. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}_{2\times 3}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$AX = b$ is inconsistent, because

rank $A = 1 \neq 2 = \text{rank}[A|b]$.

c) If the system $Ax = 0$ has a non-trivial solution, then it has infinitely many non-trivial solutions.

true. If $AX = 0$ has a nontrivial solution $X_0 \neq 0$

then it has infinitely many non-trivial solutions $X = tX_0$, ($t \in R$, $t \neq 0$).

6d) If $AB = BA$ then either A or B is a multiple of the identity.

false. e.g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = B$,

then $AB = BA$.

e) If $AB = 0$ then $A = 0$ or $B = 0$.

false. e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$,

then $AB = 0$, but $A \neq 0$, $B \neq 0$.

f) If B has a column of zeros then AB has a column of zeros.

true. If the ith column of B is B_i, and $B_i = 0$. Then the ith column of AB is $AB_i = A0 = 0$.