MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 10
15.3 #8,16; 15.4 #2,12

Problem 15.3 #8 The function

$$f(z) = \frac{z^4}{z^2 - iz + 2} = \frac{z^4}{(z-2i)(z+i)}$$

has singularities (simple poles) at $z = 2i$ and $z = -i$. The residues are given by formula (4), p.783 (or "method 4" from class):

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} = \frac{z_0^4}{2z_0 - i}$$

Therefore, we obtain:

$$\operatorname*{Res}_{z=2i} f(z) = \frac{(2i)^4}{2(2i) - i} = \frac{16}{3i} = -\frac{16}{3}i$$

$$\operatorname*{Res}_{z=-i} f(z) = \frac{(-i)^4}{2(-i) - i} = \frac{1}{-3i} = \frac{1}{3}i$$

Problem 15.3 #16 The singularities of

$$f(z) = \frac{e^z + z}{z^3 - z} = \frac{e^z + z}{z(z+1)(z-1)}$$

are at $z = 0$, $z = -1$, and $z = 1$. The path C is the counterclockwise circle centered at 0 or radius $\frac{1}{2}\pi \approx 1.5$. This path contains all three singularities. We therefore need to determine their residues. Using formula (4), we get

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} = \frac{e^{z_0} + z_0}{3z_0^2 - 1}$$

Therefore, the three residues are:

$$\begin{aligned} \operatorname{Res}_{z=0} f(z) &= \tfrac{1}{-1} = -1 \\ \operatorname{Res}_{z=-1} f(z) &= \tfrac{e^{-1}-1}{2} \\ \operatorname{Res}_{z=1} f(z) &= \tfrac{e+1}{2} \end{aligned}$$

The integral is:

$$\oint_C \frac{e^z + z}{z^3 - z} dz = 2\pi i\left(-1 + \frac{e^{-1} - 1}{2} + \frac{e+1}{2}\right) = \pi i(-2 + e^{-1} + e)$$

Problem 15.4 #2 To solve

$$\int_0^{2\pi} \frac{1}{25 - 24\cos\theta} d\theta,$$

we do the substitution $z = e^{i\theta}$. Note that as θ runs from 0 to 2π, z will traverse the counterclockwise unit circle C. Further,

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + \frac{1}{z})$$

and $dz/d\theta = ie^{i\theta} = iz$. Therefore

$$\int_0^{2\pi} \frac{1}{25 - 24\cos\theta} d\theta = \oint_C \frac{1}{25 - 24(\frac{1}{2}(z + \frac{1}{z}))} \frac{dz}{iz}$$

$$= -\frac{1}{i} \oint_C \frac{1}{12z^2 - 25z + 12} dz$$

To find the singularities, we use the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{25 \pm \sqrt{625 - 576}}{24} = \frac{25 \pm \sqrt{49}}{24},$$

so the two singularities are $z_0 = 32/24 = 4/3$ and $z_1 = 18/24 = 3/4$. Of these, only z_1 is inside the path of integration. The residue at z_1 is:

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{1}{24z_1 - 25} = \frac{1}{18 - 25} = -\frac{1}{7},$$

where $f(z) = \frac{1}{12z^2 - 25z + 12}$. Therefore, the integral is

$$-\frac{1}{i} \oint_C \frac{1}{12z^2 - 25z + 12} dz = -\frac{1}{i} 2\pi i(-\frac{1}{7}) = \frac{2\pi}{7}.$$

Problem 15.4 #12 We solve

$$\int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^3} dx$$

by the residue method. Note that the denominator is of high enough degree for this method to be applicable. The singularities are at $\pm i$, and they are both poles of order 3. The residue at $z_0 = i$ is obtained using formula (5) (or "method 5" from class):

$$\begin{aligned}
\mathrm{Res}_{z=z_0} f(z) &= \tfrac{1}{2}\lim_{z\to z_0} \tfrac{d^2}{dz^2}((z-i)^3 f(z)) \\
&= \tfrac{1}{2}\lim_{z\to z_0} \tfrac{d^2}{dz^2}(z+i)^{-3} \\
&= \tfrac{1}{2}\lim_{z\to z_0} 12(z+i)^{-5} \\
&= \tfrac{12}{64i} = -\tfrac{3i}{16}.
\end{aligned}$$

Therefore, the desired integral equals:

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}dx = 2\pi i \sum \mathrm{Res}\,\frac{1}{(1+z^2)^3} = 2\pi i(-\frac{3i}{16}) = \frac{3\pi}{8}.$$