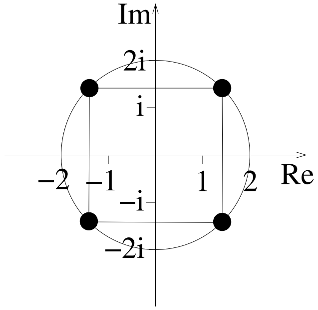
MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 2
12.2 #24,28; 12.5 #6,10; 12.6 #18

Problem 12.2 #24 To calculate $\sqrt[4]{-4}$, we first express $z = -4$ in polar coordinates. We have $r = |z| = 4$ and $\theta = \operatorname{Arg} z = \pi$, hence $z = re^{i\theta} = 4e^{i\pi}$.

The 4th roots will be of the form $w = Re^{i\varphi}$, where $R = \sqrt[4]{r} = \sqrt{2}$ and $\varphi = \pi/4 + 2\pi k/4$, where k is any integer. Thus, we have:



For $k = 0$: $\quad w = \sqrt{2}e^{\pi/4} = 1 + i$
For $k = 1$: $\quad w = \sqrt{2}e^{3\pi/4} = -1 + i$
For $k = 2$: $\quad w = \sqrt{2}e^{5\pi/4} = -1 - i$
For $k = 3$: $\quad w = \sqrt{2}e^{7\pi/4} = 1 - i$

These are the four 4th roots of z.

Problem 12.2 #28 Recall that a quadratic equation $az^2 + bz + c = 0$ is solved by the quadratic formula:

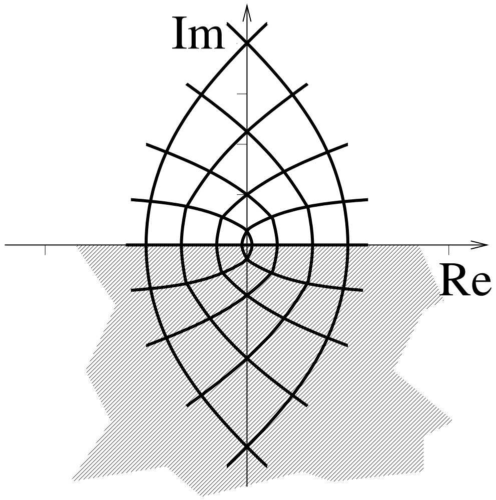
$$z_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The same holds true when a, b, c and z are complex numbers. Thus, to solve $z^2 - (5 + i)z + 8 + i = 0$, we have $a = 1$, $b = -5 - i$, and $c = 8 + i$, hence

$$\begin{aligned} z_{1/2} &= \tfrac{5+i\pm\sqrt{(-5-i)^2-4(8+i)}}{2} \\ &= \tfrac{5+i\pm\sqrt{25+10i-1-32-4i}}{2} \\ &= \tfrac{5+i\pm\sqrt{-8+6i}}{2}. \end{aligned}$$

By using the same method as in Problem #24, we find that the two square roots of $-8 + 6i$ are $\pm(1 + 3i)$, and therefore the two answers are:

$$\begin{aligned} z_1 &= \tfrac{5+i+(1+3i)}{2} = 3 + 2i \\ z_2 &= \tfrac{5+i-(1+3i)}{2} = 2 - i \end{aligned}$$



Problem 12.5 #6 The region R in the z-plane is given by $x > 0$, $y < 0$. This is the open 4th quadrant of the plane. In polar coordinates, it corresponds to $r > 0$, $-\frac{\pi}{2} < \theta < 0$, where $z = re^{i\theta} = x + iy$. Under the function $w = z^2$, we get $w = Re^{i\varphi} = r^2e^{2i\theta}$, where $R = r^2 > 0$ and $-\pi < \varphi = 2\theta < 0$. Thus, the angles are doubled. The image of the region R under $f(z) = z^2$ is therefore the open 3rd and 4th quadrant, in other words, $y < 0$.

Problem 12.5 #10 The constraint is $x \geqslant 1$, and we have $w = 1/z$, where $z = x + iy$ and $w = u + iv$. We want to express the constraint $x \geqslant 1$ in terms of u and v. We have:

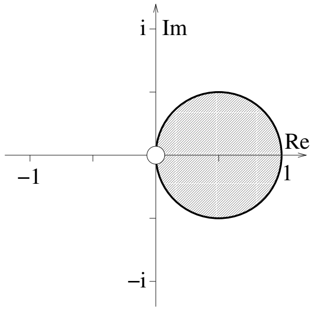
$$x + iy = z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2} = \frac{u}{u^2 + v^2} + i\frac{-v}{u^2 + v^2},$$

therefore

$$x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}.$$

We therefore have (assuming that $u^2 + v^2 \neq 0$):

$$\begin{aligned} x \geqslant 1 \quad &\Longleftrightarrow \quad \tfrac{u}{u^2+v^2} \geqslant 1 \\ &\Longleftrightarrow \quad u \geqslant u^2 + v^2 \\ &\Longleftrightarrow \quad u^2 - u + v^2 \leqslant 0 \\ &\Longleftrightarrow \quad u^2 - u + \tfrac{1}{4} + v^2 \leqslant \tfrac{1}{4} \\ &\Longleftrightarrow \quad (u - \tfrac{1}{2})^2 + v^2 \leqslant \tfrac{1}{4}. \end{aligned}$$



This is the equation of a closed disc of radius $\frac{1}{2}$ centered at $(u, v) = (\frac{1}{2}, 0)$. From the closed disc, we further have to remove the point $(0, 0)$, since x is undefined when $(u, v) = (0, 0)$.

Problem 12.6 #18 We are given $w = e^z$, where $\pi < y \leqslant 3\pi$ and $z = x + iy$. We are supposed to sketch the image of this region in the w-plane. Writing $w = Re^{i\varphi}$

in polar coordinates, we have $w = e^z = e^{x+iy} = e^x e^{iy}$, hence $R = e^x$ and $\varphi = y$. Since the is no constraint on x, R can be any value with $R > 0$. Further, the argument $\theta = y$ is between π and 3π, which is a full turn. The image is therefore the entire w-plane, minus the origin.