MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 4
12.4 #18,22,26; 12.8 #6,20

Problem 12.4 #18 The function $u = x^2 + y^2$ satisfies: $u_x = 2x$, $u_{xx} = 2$, $u_y = 2y$, $u_{yy} = 2$, $u_{xx} + u_{yy} = 4$. Therefore, it is not harmonic.

Problem 12.4 #22 The function $u = \sin x \cosh y$ satisfies: $u_x = \cos x \cosh y$, $u_{xx} = -\sin x \cosh y$, and $u_y = \sin x \sinh y$, $u_{yy} = \sin x \cosh y$, thus $u_{xx} + u_{yy} = 0$. Therefore, this function is harmonic.

The conjugate harmonic is a function $v(x, y)$ satisfying $v_y = u_x = \cos x \cosh y$ and $v_x = -u_y = -\sin x \sinh y$. Integrating the first equation with respect to y, we get:

$$v = \int \cos x \cosh y \, dy = \cos x \sinh y + C(x)$$

Plugging this into the second equation, we get $v_x = -\sin x \sinh y + \frac{d}{dx}C(x) = -\sin x \sinh y$, therefore $\frac{d}{dx}C(x) = 0$, hence $C(x) = K = const$. Therefore, $v = \cos x \sinh y + K$.

Finally, u and v together define an analytic function

$$f(z) = u(x, y) + iv(x, y) = \sin x \cosh y + i(\cos x \sinh y + K).$$

Problem 12.4 #26 Consider $u = ax^3 + bxy$. Then $u_x = 3ax^2 + by$, $u_{xx} = 6ax$, $u_y = bx$, $u_{yy} = 0$. This is analytic if $u_{xx} + u_{yy} = 6ax \equiv 0$ for all x, which is the case if and only if $a = 0$.

In case $a = 0$, we have $u = bxy$, and we calculate the conjugate harmonic $v(x, y)$ such that $v_y = u_x = by$ and $v_x = -u_y = -bx$. Integrating the first equation, we get

$$v = \int by \, dy = \frac{b}{2}y^2 + C(x)$$

Plugging this into the second equation, we get $v_x = \frac{d}{dx}C(x) = -bx$, therefore $C(x) = -\frac{b}{2}x^2 + K$, where K is a constant. Therefore, $v = \frac{b}{2}y^2 - \frac{b}{2}x^2 + K$ is the conjugate harmonic.

Problem 12.8 #6 We first write $z = -12 - 16i$ in polar coordinates. We have $z = re^{i\theta}$ where $r = |z| = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$ and $\theta = -\pi + \arctan(-16/-12) \approx -2.2142974$. Remember to subtrace π because the number $z = -12 - 16i$ is in the 3rd quadrant of the complex plane.

Then $\mathrm{Ln}(z) = \mathrm{Ln}(-12 - 16i) \approx \mathrm{Ln}(20e^{-2.2142974i}) = \ln 20 - 2.2142974i \approx 2.9957323 - 2.2142974i$.

Problem 12.8 #20 By convention, $(2i)^{2i} = e^{2i\,\mathrm{Ln}(2i)}$. To calculate $\mathrm{Ln}(2i)$, we express $2i$ in polar coordinates:

$$\mathrm{Ln}(2i) = \mathrm{Ln}(2e^{i\pi/2}) = \ln 2 + i\pi/2.$$

Therefore

$$\begin{aligned} (2i)^{2i} &= e^{2i\,\mathrm{Ln}(2i)} \\ &= e^{2i(\ln 2 + i\pi/2)} \\ &= e^{-\pi + 2i\ln 2} \\ &= e^{-\pi}(cos(2\ln 2) + i\sin(2\ln 2)). \end{aligned}$$

This expression can be evaluated with a (real number) calculator:

$$\begin{aligned} (2i)^{2i} &= e^{-\pi}(cos(2\ln 2) + i\sin(2\ln 2)) \\ &\approx 0.043213918(0.18345697 + 0.98302774i) \\ &\approx 0.0079278947 + 0.0424804804i \end{aligned}$$